Kinetics models of particles interacting with their environment

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Modèles cinétiques de particules en interaction avec leur environnement

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Modèles cinetiques de particules en interaction avec leur environnement

Résumé : Dans cette thèse, nous étudions la généralisation à une infinité de particules d'un modèle hamiltonien décrivant les interactions entre une particule et son environement. Le milieu est considéré comme une superposition continue de membranes vibrantes. Au bout d'un certain temps, tout se passe comme si la particule était soumise à une force de frottement linéaire. Les équations obtenus pour un grand nombre de particules sont proches des équations de Vlasov. Dans un premier chapitre, on montre d'abord l'existence et l'unicité des solutions puis on s'intéresse à certains régimes asymptotiques ; en faisant tendre la vitesse des ondes dans le milieu vers l'infini et en redimensionnant les échelles, on obtient à la limite une équation de Vlasov, on montre que si l'on modifie en plus une fonction paramètrisant le système, on obtient l'équation de Vlasov-Poisson attractive. Dans un deuxième chapitre, on ajoute un terme de diffusion à l'équation. Cela correspond à prendre en compte une agitation brownienne et un frottement linéaire sur les particules. Le principal résultat de ce chapitre est la convergence de la distribution de particules vers une unique distribution stationnaire. On montre la limite de diffusion pour ce nouveau système en faisant tendre simultanément la vitesse de propagation vers l'infini. On obtient une équation plus simple pour la densité spatiale. Dans le chapitre 3, nous montrons la validité des équations déjà étudiées par une limite de champ moyen. Dans le dernier chapitre, on étudie l'asymptotique en temps long de l'équation décrivant l'évolution de la densité spatiale obtenue dans le chapitre 2, des résultats faibles de convergence sont obtenus.

Kinetics models of particles interacting with their environment

Abstract : The goal of this PhD is to study a generalisation of a model describing the interaction between a single particle and its environment. We consider an infinite number of particles represented by their distribution function. The environment is modelled by a vibrating scalar field which exchanges energy with the particles. In the single particle case, after a large time, the particle behaves as if it were subjected to a linear friction force driven by the environment. The equations that we obtain for a large number of particles are close to the Vlasov equation. In the first chapter, we prove that our new system has a unique solution. We then care about some asymptotic issues ; if the wave velocity in the medium goes to infinity, adapting the scaling of the interaction, we connect our system with the Vlasov equation. Changing also continuously a function that parametrizes the model, we also connect our model with the attractive Vlasov-Poisson equation. In the second chapter, we add a diffusive term in our equation. It means that we consider that the particles are subjected to a friction force and a Brownian motion. Our main result states that the distribution function converges to the unique equilibrium distribution of the system. We also establish the diffusive limit making the wave velocity go to infinity at the same time. We find a simpler equation

satisfied by the spatial density. In chapter 3, we prove the validity of both equations studied in the two first chapters by a mean field limit. The last chapter is devoted to studying the large time asymptotic properties of the equation that we obtained on the spatial density in chapter 2. We prove some weak convergence results.

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Chapitre 1

Introduction

L'objectif de cette thèse est d'étudier la généralisation à un grand nombre de particules d'un modèle initialement proposé par Stephan de Bièvre et Laurent Bruneau pour décrire les interactions entre une particule et le milieu dans lequel elle se déplace.

1.1 Le modèle de Laurent Bruneau et Stephan de Bièvre pour une particule

Dans ce premier modèle étudié dans [17], le milieu est modélisé par une succession de membranes situées en tous points de l'espace. Ces membranes vibrent dans une direction transverse à l'espace dans lequelle évolue la particule. Cette direction transverse peut représenter les différentes variables locales libres décrivant le système. En notant t, le temps, $x \in \mathbb{R}^d$ la variable d'espace et $y \in \mathbb{R}^n$ la variable transverse, l'état du milieu au temps t et à la position (x, y) est représenté par une quantité $\Psi(t, x, y) \in \mathbb{R}$ tandis que la position de la particule au temps t est repérée par $q(t) \in \mathbb{R}^d$. L'évolution de ces deux quantités du système est couplée selon les équations d'évolution suivantes

$$\begin{cases} m\ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(q(t) - z)\ \sigma_2(y)\ \nabla_x \Psi(t, z, y)\, \mathrm{d}y\, \mathrm{d}z, \\ \partial_{tt}^2 \Psi(t, x, y) - c^2 \Delta_y \Psi(t, x, y) = -\sigma_2(y)\sigma_1(x - q(t)), \qquad x \in \mathbb{R}^d,\ y \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

où les fonctions de forme $\sigma_1 \in C_c^\infty(\mathbb{R}^d)$ et $\sigma_2 \in C_c^\infty(\mathbb{R}^n)$ sont deux fonctions positives radiales qui déterminent le couplage entre la trajectoire de la particule et les vibrations du milieu. V désigne un potentiel extérieur auquel est soumise la particule et c est la vitesse de déplacement des ondes dans le milieu. Le système est complété par les données initiales

$$q(0) = q_0, \qquad \dot{q}(0) = p_0, \qquad \Psi(0, x, y) = \Psi_0(x, y), \qquad \partial_t \Psi(0, x, y) = \Psi_1(x, ys'exprimant à partir). \tag{1.2}$$

On peux vérifier que la quantité suivante

$$E = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\partial_t \Psi(t, x, y)|^2\ \mathrm{d}x\, \mathrm{d}y + \frac{c^2}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\nabla_y \Psi(t, x, y)|^2\ \mathrm{d}x\, \mathrm{d}y + \frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) + \Phi(q(t))$$

est conservée au cours du temps. De gauche à droite, E s'interprète comme la somme d'une énergie de déplacement et d'une énergie élastique pour le milieu et d'une énergie cinétique et d'une énergie potentielle pour la particule (le terme $\Phi(q)$ se lisant comme une énergie potentielle élastique d'intéraction entre la particule et le milieu) . L'intérêt majeur de ce modèle réside dans les résultats asymptotiques suivants (valables pour $n = 3$ lorsque c satisfait une condition de grandeur dépendant d'un paramètre $\eta > 0$) que nous donnons sans détailler davantage les hypothèses de validité (voir [17, Th. 2-4]).

On se donne $\eta \in]0, 1[$, il existe une constante γ ne dépendant que des fonctions de forme σ_1 et σ_2 tel que

- Lorsque la particule est soumise à une force constante pas trop grande F ($V(x) = F \cdot x$), on peut montrer qu'il existe $q_\infty \in \mathbb{R}^d$ et une vitesse limite $v(F)$ tels que

  $$|q_\infty + tv(F) - q(t)| \leq Ke^{(1-\eta)\gamma t}.$$

  On a de plus $v(F) \underset{c \to \infty}{\sim} \dfrac{F}{\gamma}$.

- Lorsque $V = 0$, on peut montrer qu'il existe $q_\infty \in \mathbb{R}^d$ tel que

  $$|q(t) - q_\infty| \leq Ke^{(1-\eta)\gamma t}.$$

- Lorsque V est un potentiel de confinement, alors $\dot{q}(t)$ tend vers 0 tandis que $q(t)$ converge vers un point critique de V. Lorsque ce point critique q^* est un minimum non dégénéré de V, on a en plus

  $$|q(t) - q^*| \leq Ke^{(1-\eta)\gamma t/2}$$

On rappelle qu'un potentiel de confinement est un potentiel V tel que

$$V(q) \xrightarrow[|q| \to +\infty]{} +\infty.$$

En d'autres termes, lorsque la vitesse de déplacement des ondes dans le milieu est suffisamment grande, au bout d'un certain temps tout se passe comme si le milieu exerçait sur la particule une force de frottement linéaire non conservative $F = -\gamma \dot{q}(t)$. Le comportement de q est le même que celui des solutions de

$$\ddot{q} = -\gamma \dot{q} + \nabla V(q). \tag{1.3}$$

Cette équation apparaît dans de nombreux systèmes physiques, c'est l'équation du mouvement d'une petite particule dans un milieu visqueux, c'est également celle d'un electron dans un matériau conducteur selon le modèle de Drude destiné à expliquer la lois d'Ohm. Physiquement, (1.3) décrit le mouvement d'une particule considérée comme un système ouvert dans le sens qu'elle perd une énergie au profit d'un milieu dont l'état n'est pas modifié de manière apparente pour la particule par son passage. Le modèle (1.1) que nous venons de présenter s'intègre dans une grande famille de modèles hamiltoniens destinés à "fermer" le système en redistribuant l'énergie perdue par la particule dans certaines variables internes du milieu. Les premiers modèles de ce genre avaient pour objectif de justifier l'équation de Langevin qui est légèrement différente

$$m\ddot{q}(t) + \int_{\infty}^{1} \gamma(t-s)\dot{q}(s)\,\mathrm{d}s = -\nabla V(q(t)) + F_L(t).$$

Tous ces modèle présentent les similitudes suivantes :

- Le milieu est considéré comme un "bain d'oscillateurs" : les variables internes du milieu sont modélisées par des oscillateurs. Ces derniers peuvent être localisés en certains points de l'espace ou non, en nombre finis ou infinis, vibrant à des fréquences imposées (oscillateurs harmoniques) ou libres comme ici.

- Au cours de son mouvement, la particule échange de l'énergie avec les différents oscillateurs avec lesquelles elle interagit.

- Le milieu a une énergie interne dont l'expression ne fait pas intervenir la particule. En plus de l'énergie potentielle due au potentiel extérieur V et de l'énergie cinétique, la particule a également une "énergie élastique" d'interaction avec le milieu. La somme de toutes ces énergies est conservée au cours du temps.

Globalement, l'influence du milieu sur la particule sera double : d'un coté il absorbe son énergie lorsqu'elle se déplace suffisamment vite, de l'autre il empêche son repos complet par ses oscillations interne. Ce dernier point sera négligeable ici. En notant (A, μ), un espace mesurable l'hamiltonien se mettra sous la forme générale suivante

$$\begin{aligned} H_A(p,q,\pi,\phi) = \ & \frac{|p|^2}{2m} + V(q) + \int_A \left[\frac{|\pi(\alpha)|^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 |\phi_\alpha|^2 \right] \mathrm{d}\mu(\alpha) \\ & + \int_A \sigma_\alpha(q)\phi_\alpha \,\mathrm{d}\mu(\alpha) + \int_A W_\alpha(q)\,\mathrm{d}\mu(\alpha) \end{aligned}$$

En prenant $A = \mathbb{R}_x^d \times \mathbb{R}_\xi^n$, $\mathrm{d}\mu = \mathrm{d}x\,\mathrm{d}\xi$, en notant $\widehat{f}(x,\xi)$, la transformée de Fourier partielle de f par rapport à la variable transverse y pour tout f tel que $f(x,\cdot) \in L^2(\mathbb{R}_y^n)$, on retrouve bien notre modèle pour $\sigma_\alpha(q) = \sigma_1(x-q)\widehat{\sigma_2}(\xi)$, $\omega_{(x,\xi)} = c|\xi|$, $m_\alpha = m = 1$ et $W = 0$. Le couple (π, ϕ) va représenter $(\widehat{\partial_t \Psi}, \widehat{\Psi})$. La méthode de résolution de ces modèles est en générale la suivante : comme les équations décrivant le milieu sont linéaires, il est possible de les résoudre explicitement en fonction de la trajectoire de la particule et des données initiales. On obtient ensuite une équation de mouvement pour la particule. Le milieu exerce d'un coté une force de fluctuation due à son état initial (il n'y a jamais d'amortissement dans les oscillateurs puisque l'objectif est de décrire une évolution conservant l'énergie) et de l'autre une force dépendant de toutes les positions occupées par la particule aux temps passés lorsqu'elle a excité le milieu.

A titre d'exemple, on pourra trouver dans [59, 27, 26], la description du cas où la particule intéragit avec des oscillateurs harmoniques localisés en plusieurs points de l'espace. On verra dans [28] le cas d'une particule intéragissant avec un seul oscillateur. Les occillateurs peuvent être couplés comme dans [39] se combiner en chaine comme c'est le cas dans [29]. On peut également considérer que la particule reste reliée à un grand nombre d'occillateurs tout au long de son mouvement comme dans [40], ect...

Au sein de cette grande famille, le modèle (1.1) étudié dans [17] présente la double particularité d'être spatialement homogène (on entend par là que le milieu est invariant par translation dans l'espace $\mathbb{R}^d$) et de s'approcher du modèle de friction (1.3). On n'a pas encore trouvé d'autres modèles ayants ces deux propriétés d'où son intérêt particulier.

Avant de finir la description de cette famille de modèle, il est légitime de se demander pourquoi l'on fait se propager les ondes selon une direction transverse et non dans l'espace dans lesquelles se déplacent les particules. En fait, si on ne change que cela à (1.1), on obtient le système suivant.

$$\begin{cases} \ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\int_{\mathbb{R}^d} \sigma_1(q(t)-z)\ \nabla_x \Psi(t,z)\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x) - c^2 \Delta_x \Psi(t,x) = -\sigma_1(x-q(t)), \qquad x \in \mathbb{R}^d. \end{cases}$$

Le comportement asymptotique étudié dans [56, 57, 55] est en fait très différent ; lorsque $V = 0$, la vitesse de la particule converge bien mais plus nécéssairement vers 0. La force de réaction $F(v)$ du milieu à une certaine vitesse v (qui est de l'ordre de $-\gamma v$ dans (1.1)) est identiquement nulle pour toute vitesse $|v| \leq c$. Dans le cas où V est un potentiel de confinement, $q(t)$ converge bien vers un point critique du potentiel et à vitesse exponentielle lorsque ce point est un minimum non dégénéré mais contrairement à ce qu'on observe pour (1.1), la vitesse de convergence dépend a priori du potentiel ce qui n'est pas le cas pour des modèles de type (1.3).

On peu également rapprocher ce type de modèle des gaz de Lorenz où au cours de son mouvement, la particule rencontre des obstacles placés de manière déterministe ou aléatoire, on peut regarder [10, 18, 42, 44, 67, 1] pour avoir une idée des travaux contemporains dans ce domaine. Les obstacles peuvent être considérés comme mous (dans ce cas, ils sont représentés par un potentiel à support compact agissant sur la particule) ou durs (dans ce cas, la particule obéit à des conditions de réflexion lorsqu'elle rencontre les obstacles).

1.2 Généralisation à plusieurs particules

Nous allons maintenant généraliser (1.1) à une densité continue de particules, on peut voir cette généralisation de deux manières différentes, l'une probabiliste et l'autre déterministe. Nous allons d'abord considérer cette dernière qui nous semble plus naturelle. En premier lieu, nous allons maintenant montrer comment on peut établir de manière intuitive les nouvelles équations d'évolution. On présentera ensuite les liens rigoureux qui relient directement ces équations au modèle (1.1).

Formalisme cinétique

Introduction des équations

Equation d'évolution pour les particules

Étant donné un grand nombre de particules se promenant dans l'espace, nous notons f, leur fonction de distribution en espace et en vitesses ; c'est à dire qu'étant donnés Ω_1 et Ω_2, deux ensembles mesurables de $\mathbb{R}^d$,

$$\int_{\Omega_1\times\Omega_2} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \tag{1.4}$$

est la masse des particules positionnées dans Ω_1 dont les vitesses sont dans Ω_2. En supposant que les particules interagissent chacune avec le milieu suivant (1.1), leur trajectoires (p,q) sont régies par l'équation différentielle suivante

$$\left\{\begin{array}{l} \dot{q}(t) = p(t), \\ \dot{p}(t) = -\nabla V(q(t)) - \nabla\Phi(t,q(t)) \end{array}\right. \tag{1.5}$$

où Φ est donné par

$$\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \Psi(t,z,y)\sigma_1(x-z)\sigma_2(y)\,\mathrm{d}z\,\mathrm{d}y. \tag{1.6}$$

Sous réserve que les solutions de (1.5) n'explosent pas en temps fini, en notant $\varphi_\alpha^\beta(q_0,p_0)$ l'unique solution de (1.5) au temps β pour la donnée initiale $(q(\alpha),p(\alpha)) = (q_0,p_0)$, la distribution f doit satisfaire

$$f(t,x,v) = f(0,\varphi_t^0(x,v)). \tag{1.7}$$

En différenciant cette équation par rapport au temps pour une donnée initiale f_0 suffisament régulière, on déduit que f est solution de l'équation de transport suivante

$$\left\{\begin{array}{l} \partial_t f + v.\nabla_x f - \nabla_v f.\nabla_x(V+\Phi) = 0, \\ f(0,x,v) = f_0(x,v). \end{array}\right. \tag{1.8}$$

Cette équation appartenant à la famille des équations cinétiques non collisionnelles présente déjà de remarquables proprités. En supposant Φ et V connus, suffisament réguliers et minorés, il n'est pas difficile de montrer que toutes les solutions de (1.8) s'écrivent sous la forme (1.7) lorsque φ est le flot associé à (1.5). En regardant (1.5) comme l'équation différentielle sur $\mathbb{R}^{d+d}$ associée au champ de vecteurs $X(q,p) = (p, -\nabla(V+\Phi(t))(q))$, on peut d'abord remarquer que la divergence de X est nulle puis en déduire que toutes les applications φ_α^β préservent le volume dans $\mathbb{R}^d\times\mathbb{R}^d$. Pour toute fonction suffisament régulière ω s'annulant en 0, on déduit par changement de variable une grande famille de quantités conservées :

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \omega(f(t,x,v))\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} \omega(f_0(\varphi_t^0(x,v)))\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} \omega(f_0(x,v))\,\mathrm{d}x\,\mathrm{d}v$$

- Avec $\omega(u) = \max(-u,0)$, on déduit la préservation de la positivité.

- Avec $\omega(u) = u$, on déduit la conservation de la masse totale du système.
- Avec $\omega(u) = |u|^p$, on déduit la conservation des normes L^p lorsqu'elles sont définies pour f_0 pour tout $p \in [1, +\infty[$.
- La conservation de la norme infinie lorsqu'elle est définie pour f_0, découle elle aussi de (1.7) et de la préservation du volume par le flot.

On revient maintenant à la généralisation de (1.1).

Equation d'évolution pour le milieu

A ce stade, le problème n'est pas fermé, les équations (1.6) et (1.8) ne permettent pas de dire comment évolue Ψ. Dans (1.1), l'excitation du milieu par la particule ne dépend pas de sa vitesse, ici elle ne va donc dépendre que de la densité spatiale de particules

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v. \tag{1.9}$$

Pour établir l'équation satisfaite par Ψ, on somme les contributions de chaque particule en supposant que leurs actions sur le milieu se combinent de manière linéaire. On trouve

$$\Big(\partial_{tt}^2\Psi - c^2\Delta_y\Psi\Big)(t,x,y) = -\sigma_2(y)\int_{\mathbb{R}^d}\sigma_1(x-z)\rho(t,z)\,\mathrm{d}z, \quad t \geq 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^n. \tag{1.10}$$

En supposant ρ connue, cette équation est elle aussi fermée si on ajoute les données initiales

$$\Psi(0,x,y) = \Psi_0(x,y), \qquad \partial_t\Psi(0,x,y) = \Psi_1(x,y). \tag{1.11}$$

Le système complet

Avec (1.6) et (1.8)-(1.11), le système est bien fermé. Plus précisément, en supposant que le potentiel V vérifie :

$$V(x) \geq -C(1+|x|^2)$$

pour une certaine constante $C > 0$, on montrera dans le prochain chapitre qu'il existe un poid $p_t(x,v) \geq 1$ dépendant de V, $\|\sigma_1\|_{W^{3,2}(\mathbb{R}^d)}$, $\|\sigma_2\|_{L^2(\mathbb{R}^n)}$, $\|\Psi_0\|_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}$, $\|\Psi_1\|_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}$, C et $R_0 > 0$ tel que :

Théorème 1.2.1 *Problème de Cauchy*

1. *Si $\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \leq R_0$ et $f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d, p_T(x,v)\,\mathrm{d}x\,\mathrm{d}v)$, alors il existe une unique solution faible f au problème dans $\mathcal{C}([0,T], L^1(\mathbb{R}^d\times\mathbb{R}^d))$. Cette solution est continue par rapport à f_0.*
2. *Si on a seulement $f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$, alors il existe une solution faible au problème dans $\mathcal{C}(\mathbb{R}_+, L^1(\mathbb{R}^d\times\mathbb{R}^d))$ (on n'a plus forcément unicité).*

On peut par ailleurs vérifier (d'abord formellement) que l'énergie totale du système

$$\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t\Psi(t,x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{c^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_y\Psi(t,x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\left(\frac{|v|^2}{2} + V(x) + \Phi(t,x)\right)\mathrm{d}x\,\mathrm{d}v$$

est conservée, on verra que c'est vrai en toute généralité sur les solutions dont l'existence est assurée par ce dernier théorème. Si on ne s'interesse qu'à l'évolution de f, en résolvant les équations (1.10)(1.11), on verra que l'interaction des particules entre elles via le milieu peut-être décrite directement par l'équation suivante,

$$\begin{cases} \partial_t f + v.\nabla_x f - \nabla_v f.\nabla_x\left(V + \Phi_0(t) + \int_0^t p(t-s)\Sigma * \rho(s)\,\mathrm{d}s\right) = 0, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1.12}$$

où $\Sigma = \sigma_1 * \sigma_1$, $\Phi_0 \in C^1(\mathbb{R}_+, L^\infty(\mathbb{R}^d))$ et $p \in C^1(\mathbb{R}_+,\mathbb{R})$ seront explicités plus tard. On reconnaît là une équation de Vlasov avec un potentiel d'interaction régulier dans laquelle on a rajouté une demi convolution en temps. Cette demi convolution rajoute de la régularité au système, elle nous sera utile pour optimiser les conditions d'unicité.

Justification par limite de champ moyen

La manière dont nous venons d'établir le système d'équation présente l'avantage d'être simple et intuitive mais ne décrit pas clairement comment on passe de l'échelle où quelques particules interagissent chacune de manière significative avec le milieu suivant le modèle de Stephan de Bièvre et Laurent Bruneau, et le système (1.6),(1.8)-(1.11) où les particules se trouvent cachées derrière leur densité f. Ce lien peut-être établi de manière plus rigoureuse en faisant une limite de champ moyen. On se réfère à [43] ou [79]. Tout ce qui suit sera établi rigoureusement dans le chapitre 4 de cette thèse.

Le modèle pour un nombre fini de particules

On adapte d'abord (1.1) à un nombre N de particules repérées encore dans $\mathbb{R}^d$ par leurs positions respectives $(q_j)_{1\leq j\leq N}$. On suppose d'abord que le milieu agit sur chacune d'entre elles selon (1.1) et qu'en retour, leurs actions sur le milieu se somment de manière linéaire. En définissant encore Φ par (1.6), on trouve

$$\begin{cases} m\ddot{q}_j(t) = -\nabla V(q_j(t)) - \nabla\Phi(t,q_j(t)), \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\displaystyle\sum_{k=1}^N \sigma_1(x-q_k(t)). \end{cases} \tag{1.13}$$

Le système est encore complété par les données initiales (1.11) pour le milieu et

$$(q_j(0),\dot{q}_j(0)) = (q_{0,j},p_{0,j}) \quad 1\leq j\leq N, \tag{1.14}$$

pour les particules. L'idée est ensuite de faire tendre N vers l'infini. Afin de garder une masse totale finie, on suppose que chaque particule est de masse proportionnelle à $\frac{1}{N}$. Pour que le terme source de l'équation des ondes satisfaite par Ψ n'explose pas, on le renormalise en supposant que l'action de chaque particule sur le milieu est proportionnelle à sa masse. Cela revient à multiplier ce terme source par $\frac{1}{N}$. Cette hypothèse peut également se voir comme une simple continuité ; lorsque deux particules se rapprochent au point d'occuper des positions semblables, tout se passe comme si on n'avait plus qu'une particule de masse double, on ne s'attend pas à voir l'action des deux particules sur le milieu être subitement divisée par deux lorsque leur positions coïncident. Par symétrie d'action, on multiplie également Φ par $\frac{1}{N}$ dans l'équation vérifiée par les q_j. On est alors obligé de faire de même avec le potentiel exterieur V, cela se comprend très bien lorsque le potentiel est gravitationnel, on peut trouver d'autres interprétations dans d'autres cas. Au final, on récupère les équations suivantes

$$\begin{cases} \dfrac{m}{N}\ddot{q}_j(t) = -\dfrac{1}{N}\nabla V(q_j(t)) - \dfrac{1}{N}\nabla\Phi(t, q_j(t)), \\ \partial_{tt}^2\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \dfrac{1}{N}\displaystyle\sum_{k=1}^{N}\sigma_1(x - q_k(t)). \end{cases} \tag{1.15}$$

On peut vérifier que (1.14)-(1.15) admet bien une unique solution. On définit rigoureusement la densité empirique $\widehat{\mu}^N$ qui n'est rien d'autre que la "fonction de distribution" (en l'occurence c'est une mesure) du système

$$\widehat{\mu}_t^N = \frac{m}{N}\sum_{j=1}^{N}\delta_{(q_j(t),\dot{q}_j(t))}.$$

On peut vérifier que les équations (1.6),(1.8)-(1.11) se généralisent bien au cadre des mesures et que le couple $(\widehat{\mu}^N, \Phi)$ en est l'unique solution.

Le passage à la limite

Etant donnée n'importe quelle mesure positive finie μ_0, il n'est pas difficile de trouver une famille de données initiales $(q_{j,0}^N, p_{j,0}^N)_{\substack{N\geq 1\\ 1\leq j\leq N}}$ telle que $\widehat{\mu}_0^N$ converge étroitement vers μ_0

$$\lim_{N\to\infty}\frac{m}{N}\sum_{j=1}^{N}\chi(q_{j,0}^N, p_{j,0}^N) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_0 \qquad \forall\chi\in C_b(\mathbb{R}^d\times\mathbb{R}^d).$$

En particulier toute mesure ayant une fonction de densité $f_0 \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$ par rapport à la mesure de Lebesgue peut être approximée ainsi. La validité de (1.6),(1.8)-(1.11) pour des densités continues de particules peut être donc déduite de la stabilité de ces équations. En supposant que $\widehat{\mu}_0^N$ converge étroitement vers une mesure μ_0, cette stabilité pourra être exprimée de manière plus ou moins forte par l'une des assertions suivantes.

- $\widehat{\mu}_t^N$ converge pour tout t vers l'unique solution de (1.8)-(1.11) de donnée initiale μ_0.
- On peut extraire une sous suite k_N telle que $\widehat{\mu}_t^{k_N}$ converge pour tout t vers une solution de (1.8)-(1.11) de donnée initiale μ_0.

Nous verrons que la deuxième assertion est presque toujours vérifiée sous des hypothèses très générales sur V, la première demandera en plus une hypothèse d'intégrabilité uniforme sur la famille $(\widehat{\mu}_0^N)_N$. Une autre manière de justifier (et donc d'interpréter) le système (1.6),(1.8)-(1.11) consiste à considérer les particules comme des variables aléatoires.

Interprétation probabiliste

Une fois établies les équations (1.15) pour décrire les intéractions entre les N particules et le milieu, étant donnée une condition initiale f_0 de masse totale 1 définissant ainsi une mesure de probabilité, on peut considérer les données initiales $(q_{j,0}^N, p_{j,0}^N)_{\substack{N\geq 1\\ 1\leq j\leq N}}$ comme des variables aléatoires indépendantes identiquement distribuées selon la loi f_0. En d'autres termes, on a

$$\mathbb{P}[(q_{j,0}^N, p_{j,0}^N) \in A \times B] = \int_{A\times B} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v \qquad 1 \leq j \leq N < \infty. \tag{1.16}$$

Les données initiales étant choisies aléatoirement, les solutions $(q_j^N, p_j^N)_{1\leq j\leq N}$ évoluent de manière déterministe au cours du temps selon (1.15). Comme toutes les particules évoluent de manière symétrique, elles vont partager la même loi de probabilitité $\mu_t^{(1,N)}$ au cours du temps

$$\mathbb{P}[(q_j^N(t), p_j^N(t)) \in A\times B] = \mathbb{P}[(q_1^N(t), p_1^N(t)) \in A\times B] = \int_{A\times B} \mathrm{d}\mu_t^{(1,N)} \qquad 1 \leq i, j \leq N < \infty.$$

Lorsque V est lipschitzien, on montrera (voir la section 5 du chapitre 4 en supposant que $\gamma = 0$) que quand N tend vers l'infini, $\mu^{(1,N)}$ converge faiblement vers la solution de (1.6),(1.8)-(1.11) (unique sous ces hypothèses). Selon cette idée, $f(t)$ s'interprète comme la loi de probabilité de distribution d'une particule typique du système. Le lien entre ce point de vue et le précédent (portant sur la densité empirique) peut se comprendre très simplement par un argument de type loi des grands nombres.

Lien entre les deux interprétations

Un petit calcul nous permet de relier facilement $\widehat{\mu}_0^N$ et $\mu_0^{(1,N)}$. On se donne d'abord $\chi \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, une fonction test. N étant fixé, les $(q_{j,0}^N, p_{j,0}^N)_{1\leq j\leq N}$ forment une famille de variables aléatoires indépendantes indentiquement distribuées, il en va de même pour $\left(\chi(q_{j,0}^N, p_{j,0}^N)\right)_{1\leq j\leq N}$. La lois des grands nombres nous donne immédiatement

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{j=1}^N \chi(q_{j,0}^N, p_{j,0}^N) - \mathbb{E}[\chi(q_{1,0}^N, p_{1,0}^N)]\right|\right] \leq \frac{1}{\sqrt{N}}\left(\mathbb{E}[\chi(q_{1,0}^N, p_{1,0}^N)^2] - \mathbb{E}[\chi(q_{1,0}^N, p_{1,0}^N)]^2\right)^{1/2}.$$

Le terme de gauche se réécrit très simplement en fonction de $\widehat{\mu}_0^N$ et de $\mu_0^{(1,N)}$. En majorant grossièrement le terme de droite, on déduit une première estimation de l'écart entre ces deux mesures

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}\widehat{\mu}_0^N - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}\mu_0^{(1,N)}\right|\right] \leq \frac{\|\chi\|_{L^\infty}}{\sqrt{N}}.$$

En faisant tendre N vers l'infini, ce résultat assure la convergence en loi de $\widehat{\mu}_0^N$ vers $\mu_0^{(1,N)}$. Après n'importe quel temps $t > 0$, les particules ont toutes interagi entre elles via le milieu, l'indépendance de la famille de variables aléatoires $(q_j^N(t), p_j^N(t))_{1\leq j\leq N}$ nécéssaire au calcul précédent a toutes les raisons d'être perdue. Sachant que le couplage entre le milieu et chaque particule décroit en $\frac{1}{N}$, on s'attend à ce que l'action d'une particule sur une autre via le milieu décroisse en $\frac{1}{N^2}$. Lorsque N tend vers l'infini on peut donc espérer que les trajectoires des particules redeviennent décorrélées. Cette propriété appelée propagation du chaos sera présentée plus en détails dans le chapitre 4. On verra notamment qu'elle est bien satisfaite dans notre cas et que les deux mesures $\widehat{\mu}^N$ et $\mu^{(1,N)}$ tendent à coïncider (en loi) quand N tend vers l'infini. La notion de propagation du chaos a été d'abord introduite par Mac Kean [70], elle a été largement développée depuis, il semble assez hasardeux d'envisager de montrer la convergence des mesures importantes du système lorsqu'elle n'est pas vérifiée.

Quelques résultats asymptotiques

Nous présentons dans cette dernière sous partie quelques résultats asymptotiques démontrés sur le système (1.6),(1.8)-(1.11). L'idée est ici de "faire tendre ce système vers d'autres" pour mieux le situer dans sa famille d'équations. Le sens que l'on donne a cette notion est la suivante : On fait varier continument les paramètres (ici σ_1, σ_2, V et c) en fonction d'une variable réelle ϵ. On fait en sorte que lorsque ϵ tend vers 0, on se dirige vers un certain cadre limite. On obtient ainsi une famille $(f_\epsilon, \Psi_\epsilon)_{\epsilon>0}$ constituée des solutions de (1.6),(1.8)-(1.11) pour chaque valeur de ϵ. Lorsque ϵ tends vers 0, on montre qu'on peut extraire une sous-suite ϵ_n telle que la suite $(f_{\epsilon_n})_{n\geq 0}$ converge vers la solution d'une nouvelle équation que l'on définit ensuite comme le système limite.

Vers Vlasov

Il nous a d'abord semblé intéressant de regarder ce qui se passait pour de très grandes vitesses de propagation c. Si on se contente de faire tendre c vers l'infini, on perd le couplage entre les particules et le milieu. Ce fait s'interprète comme une conséquence des effets de moyenne qui se traduisent pour toute fonction réelle χ suffisamment régulière par la convergence

$$\lim_{c\to+\infty} \int_{-\infty}^{+\infty} \chi(x) e^{icx}\,\mathrm{d}x = 0.$$

Une très grande vitesse de propagation induit des vibrations beaucoup trop rapides dans le milieu qui tendent à annuler entre elles leurs effets sur les particules beaucoup plus lentes. Pour garder un couplage intéressant, on augmente l'amplitude du terme d'intéraction. Plus précisément, lorsque $n \geq 3$, on se donne une famille $(f_\epsilon)_\epsilon$ solution de

$$\begin{cases} \partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon - \nabla_x(V+\Phi_\epsilon)\cdot\nabla_v f_\epsilon = 0, \\ \Phi_\epsilon(t,x,y) = \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^d} \Psi_\epsilon(t,z,y)\sigma_2(y)\sigma_1(x-z)\,\mathrm{d}z\,\mathrm{d}y, \\ \Big(\partial_{tt}^2 - \dfrac{1}{\epsilon}\Delta_y\Big)\Psi_\epsilon(t,x,y) = -\dfrac{1}{\epsilon}\sigma_2(y)\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma_1(x-z) f(t,z,v)\,\mathrm{d}v\,\mathrm{d}z, \end{cases}$$

Si $\|f_{0,\epsilon}\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}$, $\|f_{0,\epsilon}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}$ et l'énergie globale du système sont uniformément bornés par rapport à ϵ au temps initial alors quitte à extraire une sous-suite $(\epsilon_n)_{n\geq 0}$, f_ϵ converge faiblement vers une solution de l'équation de Vlasov :

$$\begin{cases} \partial_t f + v.\nabla_x f = \nabla_v f.\nabla_x \left(V - \kappa\Sigma * \rho\right)) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

avec

$$\kappa = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma_2}(\xi)|^2}{(2\pi)^N|\xi|^2}\,\mathrm{d}\xi$$

Vers Vlasov Poisson

On peut aussi faire varier plusieurs paramètres en même temps, pour obtenir des modèles limites plus physiques. On suppose ici $N \geq 3$ et $d = 3$. En faisant également varier soigneusement σ_1 en fonction de c, on peut obtenir l'équation de Vlasov-Poisson attractif :

$$\begin{cases} \partial_t f + v.\nabla_x f = \nabla_v f.\nabla_x \left(V + \tilde{\Phi}\right), \\ \Delta\tilde{\Phi} = \kappa\rho, \\ f(0,x,v) = f_0(x,v). \end{cases}$$

Cette équation intervient notamment pour décrire l'évolution en temps des galaxies. L'existence des solutions est montrée pour la première fois dans [7], les méthodes que nous utilisons pour faire converger les solutions sont en grande partie similaires.

Dans ces deux cas limites, les données initiales Ψ_0 et Ψ_1 du milieu n'influent finalement plus à la limite. Ce fait est lié aux propriétés dispersives de l'équation des ondes qui se traduisent ici par des inégalités de Strichartz [69].

Le problème de l'asymptotique en temps

Le comportement en temps long est encore à déterminer, la première intuition pourait-être de supposer que toutes les particules vont se comporter comme la particule unique dans [17], on peut montrer que cette intuition est généralement fausse.

Si par exemple V est un potentiel de confinement, n'admettant (pour simplifier) qu'un point singulier q^* (c'est à dire un seul q^* tel que $\nabla V(q^*) = 0$) alors par conservation de la masse et de la positivité, la convergence hypothétique de toutes les particules vers q^* et de leur vitesse vers 0, signifie exactement la convergence de f vers la mesure de Dirac $\delta_{(q^*,0)}$. Une telle convergence est impossible dès que f_0 se trouve dans un espace L^p pour $p > 1$ par conservation des normes L^p le long des solutions.

Le cas où $V = 0$ ne semble pas mieux acquis. La même intuition voudrait que la trajectoire $q(t)$ de chaque particule soit convergente tandis que sa dérivée $\dot{q}(t)$ tend vers 0. Mathématiquement, cela se traduit par l'existence d'une certaine densité limite ρ_∞ telle que $f(t)$ converge vers $\rho_\infty \otimes \delta_0$. Une telle convergence est encore impossible dès que f_0 est bornée.

Il en va de même pour le cas où $V(x) = F.x$: cette fois ci, on attendrait l'existence d'une autre densité limite ρ_∞ telle que $f(t) - \rho_\infty(x - tv(F)) \otimes \delta_{v(F)}$ tende vers 0, c'est encore impossible sous la même hypothèse sur f_0.

Pour sortir de cette apparente impasse, nous avons ajouté un nouveau terme à notre système.

1.3 En ajoutant un opérateur de Fokker-Planck

On rajoute maintenant un terme de Fokker-Planck à l'équation d'évolution de f :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x (V + \Phi(t)) \cdot \nabla_v f = \gamma \nabla_v \cdot (vf + \nabla_v f) \\ f(0,x,v) = f_0(x,v) \end{cases} \tag{1.17}$$

Les équations (1.9)-(1.11) et (1.6) qui décrivent le couplage entre le milieu et les particules restent inchangées. Nous avertissons le lecteur que le coefficient γ n'a pas de rapport avec celui déjà apparu dans (1.3) (même si les deux représentent un coefficient de frottement). Ce nouveau système s'aborde différemment du précédent. Si la masse et la positivité sont toujours conservées, il n'en va plus de même des normes L^p. La conservation de l'énergie est remplacée par la décroissance de sa somme avec l'entropie :

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} f_\epsilon \Big(\ln(f_\epsilon) + \frac{v^2}{2} + (V + \Phi_\epsilon) \Big) \, \mathrm{d}v \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} \Big(\epsilon^2 |\partial_t \Psi_\epsilon|^2 + |\nabla_z \Psi_\epsilon|^2 \Big) \, \mathrm{d}z \, \mathrm{d}x \right\} \\ = -\frac{1}{\epsilon^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \Big| 2 \nabla_v \sqrt{F_\epsilon} + v \sqrt{F_\epsilon} \Big|^2 \, \mathrm{d}v \, \mathrm{d}x. \end{aligned}$$

L'existence de solutions à équation sera démontrée dans le cadre restreint où V est lipschitzien dans le chapitre 4.

Interprétation du terme supplémentaire

Pour comprendre (1.17), on revient aux limites de champ moyen. On considère à nouveau un nombre N de particules repérées encore dans $\mathbb{R}^d$ par leurs positions respectives $(q_j)_{1 \le j \le N}$. En plus de l'interaction déjà connue entre le milieu et les particules, on suppose deux autres choses.

- Le milieu exerce sur les particules une force de frottement proportionnelle à leur vitesse.
- Les particules sont également soumises à l'action d'un mouvent brownien aléatoire sur les particules. Ce dernier peut être vu comme la conséquence de chocs entre elles ou avec des objets extérieurs.

L'action de ce mouvement brownien change beaucoup les choses. D'abord, l'absence presque sure de régularité des mouvements browniens ne nous permet plus de supposer les positions des particules $(q_j)_{1 \le j \le N}$ deux fois dérivables, on regardera donc les variations infinitésimales $\mathrm{d}p_j$ des moments ($p_j = \dot{q}_j$) plutôt que leurs dérivées en temps. Par ailleurs, il n'y a plus d'interprétation non probabiliste possible ; puisque les mouvement browniens sont aléatoires, les trajectoires des particules sont également des variables aléatoires.

En insérant ces deux nouveaux phénomènes dans (1.15), on récupère le système d'équation suivant

$$\begin{cases} \mathrm{d}q_i^N(t) = p_i^N(t)\,\mathrm{d}t \\ \mathrm{d}p_i^N(t) = -\nabla_x(V+\Phi(t))(q_i^N(t))\,\mathrm{d}t - \gamma p_i^N(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B_i(t), \\ \partial_{tt}^2\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\,\dfrac{1}{N}\displaystyle\sum_{k=1}^{N}\sigma_1(x-q_k(t)). \end{cases} \tag{1.18}$$

Le système est encore complété par les équations (1.6) (1.11) (1.14) et (1.16). Il existe bien une unique solution au sens où si l'on fixe les mouvement browniens $(B_i)_{1\leq i\leq N}$ et les données initiales $(q_{0,j},p_{0,j})$, le système (1.6) (1.11) (1.14) (1.18) admet une unique solution. Les $(q_j(t),p_j(t))$ ont bien un sens en tant que variables aléatoires.

Les particules évoluent encore de manière symétrique, en notant $\mu_t^{(1,N)}$, leur loi de probabilité, Lorsque V est lipschitzien, $\mu^{(1,N)}$ converge étroitement vers l'unique solution de (1.11) (1.14) (1.16) (1.18). Le chaos se propage encore ; lorsque N tend vers l'infini, la mesure empirique $\widehat{\mu}$ tend à coïncider avec $\mu^{(1,N)}$. On poura se referer à [11, 80] pour cette limite.

Asymptotique en temps

Le principal résultat établi sur (1.17) concerne l'asymptotique en temps long lorsque V est un potentiel de confinement. Dans ce cas, on montre qu'au delà d'une certaine valeur c_0, le système admet un unique état d'équilibre $(\mathscr{M}_{\mathrm{eq}},\Psi_{eq})$. Sous des hypothèses techniques qui se résument à dire que V est très confinant et que les données initiales Ψ_0,Ψ_1 ne perturbent pas trop Ψ_{eq}, on montre le retour à l'équilibre dans $L^2\left(\mathbb{R}^d\times\mathbb{R}^d;\dfrac{\mathrm{d}x\,\mathrm{d}v}{\mathscr{M}_{\mathrm{eq}}(x,v)}\right)$. En d'autre terme, on peut trouver M dépendant des données initiales et $\kappa>0$ tels que

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(t,x,v)-\mathscr{M}_{\mathrm{eq}}(x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x \leq Me^{-\kappa t}.$$

Le résultat est établi grâce à des méthodes d'hypocoercitivité que l'on peut trouver développées dans [34] et notamment appliquées à l'équation de Vlasov-Fokker-Planck très proche du système que nous étudions. L'idée originelle de ces méthodes est de modifier la norme naturelle dans l'espace $L^2\left(\mathbb{R}^d\times\mathbb{R}^d;\dfrac{\mathrm{d}x\,\mathrm{d}v}{\mathscr{M}_{\mathrm{eq}}(x,v)}\right)$ en une autre norme équivalente $\|\cdot\|_H$ pour laquelle on puisse établir l'inégalité

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f(t,x,v)-\mathscr{M}_{\mathrm{eq}}(x,v)\|_H^2 \leq -\frac{\kappa}{2}\|f(t,x,v)-\mathscr{M}_{\mathrm{eq}}(x,v)\|_H^2.$$

Cette dernière implique naturellement la convergence recherchée par le lemme de Gronwall. Dans notre cas, l'équation des ondes induit des termes de retard qui sont gérés un peu plus finement.

Limite de diffusion

Sur des systèmes tels que (1.17), lorsque ϵ tend vers 0, le régime asymptotique suivant est appelé régime de diffusion

$$\partial_t f_\epsilon + \frac{1}{\epsilon}(v \cdot \nabla_x - \nabla_x(V + \Phi_\epsilon) \cdot \nabla_v) f_\epsilon = \frac{1}{\epsilon^2}\mathrm{div}_v(v f_\epsilon + \nabla_v f_\epsilon). \tag{1.19}$$

Il est maintenant bien connu (cf [30]) que pour de tels régimes, lorsque ϵ tend vers 0, la solution tend à prendre la forme suivante

$$f_\epsilon(t,x,v) \underset{\epsilon\to 0}{\sim} \rho(t,x)\frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}.$$

Si Φ ne dépend pas de ϵ, on s'attend à ce que ρ soit solution de l'équation

$$\partial_t \rho = \mathrm{div}(\nabla\rho + \rho\nabla(V + \Phi)).$$

Dans notre cadre, lorsque ϵ tend vers 0, nous faisons également tendre la vitesse de propagation des ondes c vers l'infini tout en augmentant comme précédemment le couplage entre les particules et le milieu pour éviter que leurs interactions disparaissent.

$$\begin{cases} \partial^2_{tt}\Psi_\epsilon - \dfrac{1}{\epsilon^2}\Delta_z\Psi_\epsilon(t,x,z) = -\dfrac{1}{\epsilon^2}\sigma_2(z)\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}\sigma_1(x-y)F_\epsilon(t,y,v)\,\mathrm{d}v\,\mathrm{d}y, \\ \Phi_\epsilon(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^N}\Psi_\epsilon(t,z,y)\sigma_1(x-z)\sigma_2(y)dy \end{cases} \tag{1.20}$$

A l'aide d'estimations fines sur la fonctionnelle d'entropie-énergie et sa dérivée temporelle, en notant $M(v) = \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$, nous obtenons le théorème suivant

Théorème 1.3.1 *On suppose que* $e^{-\nu V} \in L^1(\mathbb{R}^d)$ *pour* $0 < \nu < 1/2$. *On se donne une famille* $(f_\epsilon, \Psi_\epsilon)_{\epsilon>0}$ *solution de* (1.19)(1.20) *et telle que la quantité suivante est bornée :*

$$\begin{aligned}\mathscr{K}_0 = \sup_{\epsilon>0}\Bigg\{&\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(0,x,v)\Big(1 + |\ln(f_\epsilon)(0,x,v)| + V(x) + \frac{v^2}{2}\Big)\,\mathrm{d}v\,\mathrm{d}x \\ &+\frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x\Bigg\}\end{aligned}$$

Alors on peut extraire une suite faiblement convergente dans $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$ *vers* $\rho\otimes M$ *où* ρ *est solution de :*

$$\partial_t\rho - \nabla_x \cdot (\nabla_x\rho + \rho\nabla_x(V - \Lambda\Sigma * \rho))) = 0. \tag{1.21}$$

1.4 Asymptotique en temps long sur une équation de Fokker-Planck homogène

On s'est enfin intéressé à comprendre l'asymptotique en temps long de l'équation (1.21) que l'on réécrit sous une forme simplifiée

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x (V + W * \rho))) = 0, \\ \rho(0) = \rho_0. \end{cases} \tag{1.22}$$

Cette équation admet encore une fonctionnelle d'entropie

$$\mathscr{E}(\rho) = \int_{\mathbb{R}^d} \rho(\ln(\rho) + \frac{1}{2} W * \rho + V)\,\mathrm{d}x,$$

on montre que les états d'équilibre d'entropie finie sont les points fixes de l'application $\mathcal{T}$ définie par

$$\mathcal{T} : \rho \in X \mapsto \frac{m}{Z(\rho)} e^{-V - W*\rho}, \quad Z(\rho) := \int_{\mathbb{R}^d} e^{-V(x) - W*\rho(x)}\,\mathrm{d}x.$$

Sous des propriété de convexité portant sur V et W, il est maintenant bien connu [21, 66, 22] qu'il existe un unique état d'équilibre et que la solution converge vers lui à vitesse exponentielle. Le principal résultat du dernier chapitre a pour but de décrire ce qui se passe sans cette hypothèse sur W. La stratégie a été d'utiliser $\mathscr{E}$ comme une fonctionnelle de Liapounov, on s'est donc limité à considérer des données initiales ρ_0 d'entropie finie. Afin de maximiser les informations données par sa décroissance (et sa convergence), nous avons supposé V tel la mesure $\left(\int_{\mathbb{R}^d} e^{-V(x)}\,\mathrm{d}x\right)^{-1} e^{V(x)}$ satisfasse une inégalité de Sobolev logarithmique. D'abord introduites par Gross dans [49], ces inégalités sont maintenant couramment utilisés en analyse et en probabilité tant pour déduire des propriétés asymptotiques que pour obtenir des bornes intéressantes de la décroissance de l'entropie. On se référera à [4] pour une large revue de ces inégalités et de leurs applications. Nous avons Lorsque W est pair, borné et lipschitz alors pour peu que V soit convexe à un fonction L^∞ près, on a

Théorème 1.4.1 *Asymptotique en temps long*

1. *Pour toute solution régulière ρ à (1.22), l'entropie $\mathscr{E}(\rho)$ converge vers une certaine valeur limite $\mathscr{E}^*$ quand t tend vers l'infini. L'ensemble*

$$Eq(m, \mathscr{E}^*) := \left\{ \rho_{eq} \in X \quad \middle| \quad \rho_{eq} = \mathcal{T}(\rho_{eq}), \quad \int_{\mathbb{R}^d} \rho_{eq} = m, \quad \mathscr{E}(\rho_{eq}) = \mathscr{E}^* \right\}$$

   est non vide et ρ s'en rapproche de plus en plus :

$$\|\rho(t) - \mathcal{T}(\rho(t))\|_{L^1(\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0 \quad et \quad \inf_{\rho_{eq} \in Eq(m,\mathscr{E}^*)} \|\rho(t) - \rho_{eq}\|_{L^1} \xrightarrow[t\to\infty]{} 0$$

2. *Lorsque $\alpha = m\delta_W > -1/2$, alors on peut trouver un unique état d'équilibre ρ_{eq}. Avec $\beta = \frac{1+2\alpha}{1+\alpha}$, toute solution ρ to (1.22), tend vers cet état d'équilibre à vitesse exponentielle*

$$\|\rho(t) - \rho_{eq}\|_{L^1(\mathbb{R}^d)} \leq \left(\frac{2m}{1+\alpha}\right)^{1/2} (\mathcal{E}(\rho_0) - \mathcal{E}^*)^{1/2} e^{-2\kappa\beta t}$$

où κ est une constante dépendant de V et $\|W\|_{L^\infty(\mathbb{R}^d)}$, δ_W est donné par

$$\delta_W := \inf_{\|h\|_{L^1}\leq 1} q_W(h), \quad q_W(h) := \int_{\mathbb{R}^d} hW * h \,\mathrm{d}x \geq -\|W\|_{L^\infty}\|h\|_{L^1}^2.$$

Le deuxième point n'est qu'une amélioration d'un autre déjà montré dans [5]. Si le premier point ne permet pas de conclure si $\rho(t)$ converge ou non, il permet néanmoins d'établir des critères pour cela, on verra notamment que la convergence au sens des distributions et la convergence L^1 sont ici équivalentes. Dans le cas où l'ensemble des équilibres est totalement discontinu, on peut conclure directement la convergence également. Rien n'assure cependant que cet ensemble ne contienne pas des parties homéomorphes à un cercle autour de laquelle la solution pourrait continuer de tourner de plus en plus lentement sans jamais s'arrêter. Pour répondre à ces question, nous avons entrepris (toujours à l'aide de $\mathscr{E}$) de caractériser localement la nature de ces états d'équilibres.

1.5 Perspectives

Le modèle initial développé dans [17] tire en grande partie son intérêt de son comportement en temps long. On a vu que l'évolution donnée par le système (1.6),(1.8)-(1.11) conservait trop de quantités pour que les résultats valables pour une particule puissent se généraliser à une densité continue de particules. L'équation limite intuitive dans [17]

$$\ddot{q} = V(q) - \gamma\dot{q}$$

ne peut être celle de toutes les particules. Il serait très intéressant de comprendre ce que fait le système, s'il tend à se concentrer autour des "solutions intuitives" autant qu'il lui est possible de le faire sans violer ses lois de conservation, ou s'il fait tout simplement autre chose. Certains indices tendraient à valider la première hypothèse. Ainsi, lorsque $V = 0$ et $n \geq 3$, en supposant que le couple (f, Ψ) est stationnaire, alors on peut montrer que pour une certaine valeur $\kappa > 0$, on a

$$\mathrm{supp}(f) \subset \left\{(x, v) \in \mathbb{R}^d \times \mathbb{R}^d \quad \middle| \frac{|v|^2}{2} \leq \frac{\kappa}{c^2}(\Sigma * \rho)(x)\right\}.$$

Lorsque c est grand, cela implique bien que les vitesses des particules sont petites. Tout cela n'assure pas que ces équilibres existent sans même évoquer leur possible stabilité. Dans le cas où V est un potentiel de confinement, on peut montrer l'existence d'une très grande famille d'états d'équilibre admissibles. Le choix entre chacun d'entre eux semble assez ardu en premier lieu. Dans le cas radial, la stabilité de certains états d'équilibre a été démontré pour les équations de Vlasov-Poisson dans [60, 61]. Nous avons montré que les équation de Vlasov-Poisson pouvaient être obtenues comme une limite de notre système, on peut donc espérer pouvoir adapter les méthodes qui marchent pour ces équations à notre cadre. Il serait également tentant de chercher à généraliser les résultats du dernier chapitre à des potentiels d'interaction plus physiques, ou d'aller d'abord plus loin dans la description des états d'équilibre pour ensuite comprendre ce que fait la solution à leur voisinage.

Chapitre 2

Un premier modèle cinétique : le système Vlasov-Ondes

Dans cet article écrit en collaboration avec Stephan de Bièvre et Thierry Goudon, nous étudions la généralisation du modèle d'interaction entre une particule et son milieu issu de [17] à une densité continue de particules. Nous reformulons le système naturel obtenu afin simplifier les calculs qui suivent, le système simplifié est proche des équations de type Vlasov. Nous prouvons l'existence de solutions sous des hypothèses très générales et donnons une bonne condition d'unicité de ces solutions à l'aide de méthodes déjà présentes dans [33]. Nous étudions ensuite deux régimes asymptotiques. Dans le premier régime, nous obtenons une équation de Vlasov avec un potentiel régulier. Dans le second, nous obtenons l'équation de Vlasov-Poisson attractif par des méthodes issues de [7].

2.1 Introduction

In [17], L. Bruneau and S. De Bièvre introduced a mathematical model intended to describe the interaction of a classical particle with its environment. The environment is modeled by a vibrating scalar field, and the dynamics is governed by energy exchanges between the particle and the field, embodied into a Hamiltonian structure. To be more specific on the model in [17], let us denote by $q(t) \in \mathbb{R}^d$ the position occupied by the particle at time t. The environment is represented by a field $(t, x, y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \mapsto \Psi(t, x, y) \in \mathbb{R}$: it can be thought of as an infinite set of n-dimensional membranes, one for each $x \in \mathbb{R}^d$. The displacement of the membrane positioned at $x \in \mathbb{R}^d$ is given by $y \in \mathbb{R}^n \mapsto \psi(t, x, y) \in \mathbb{R}$. The coupling is realized by means of form factor functions $x \mapsto \sigma_1(x)$ and $y \mapsto \sigma_2(y)$, which are supposed to be non-negative, infinitely smooth, radially symmetric and compactly supported. Therefore, the dynamic is described by the following set of differential equations

$$\begin{cases} \ddot{q}(t) = -\nabla V(q(t)) - \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-z)\ \sigma_2(y)\ \nabla_x\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ y\in\mathbb{R}^n. \end{cases} \tag{2.1}$$

In (2.1), $c>0$ stands for the wave speed in the transverse direction, while $q\in\mathbb{R}^d \mapsto V(q)\in\mathbb{R}$ is a time-independent external potential the particle is subjected to. In [17], the well-posedness theory for (2.1) is investigated, but the main issue addressed there is the large time behavior of the system. It is shown that the system exhibits dissipative features: under certain circumstances (roughly speaking, $n=3$ and c large enough) and for a large class of finite energy initial conditions the particle energy is evacuated in the membranes, and the environment acts with a friction force on the particle. Accordingly, the asymptotic behavior of the particle for large times can be characterized depending on the external force: if $V=0$, the particle stops exponentially fast, when V is a confining potential with a minimizor q_0, then the particle stops at the location q_0, and for $V(q)=-F\cdot q$, a limiting velocity V_F can be identified.

Since then, a series of works has been devoted to further investigation of the asymptotic properties of a family of related models. We refer the reader to [1, 26, 27, 28, 59] for thorough numerical experiments and analytical studies, that use random walks arguments in particular. The model can be seen as a variation on the Lorentz gas model where one is interested in the free motion of a single point particle in a system of obstacles distributed on a certain lattice. We refer the reader to [10, 18, 42, 44, 67] for results and recent overviews on the Lorentz gas problem. Instead of dealing with periodically or randomly distributed hard scatterers as in the Lorentz gas model, here the particle interacts with a vibrational environment, that create the "soft" potential Φ. The asymptotic analysis of the behavior of a particle subjected to an oscillating potential is a further related problem that is also worth mentioning [41, 48, 54, 75].

We wish to revisit the model of [17], in the framework of kinetic equations. Instead of considering a single particle described by its position $t\mapsto q(t)$, we work with the particle distribution function in phase space $f(t,x,v)\geq 0$, with $x\in\mathbb{R}^d$, $v\in\mathbb{R}^d$, the position and velocity variables respectively. This quantity obeys the following Vlasov equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x(V+\Phi)\cdot\nabla_v f = 0, \qquad t\geq 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d. \tag{2.2}$$

In (2.2), V stands for the external potential, while Φ is the self-constistent potential describing the interaction with the environment. It is defined by the convolution formula

$$\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \Psi(t,z,y)\sigma_1(x-z)\sigma_2(y)\,\mathrm{d}y\,\mathrm{d}z, \qquad t\geq 0,\ x\in\mathbb{R}^d \tag{2.3}$$

where the vibrating field Ψ is driven by the following wave equation

$$\begin{cases} \Big(\partial^2_{tt}\Psi - c^2\Delta_y\Psi\Big)(t,x,y) = -\sigma_2(y) \displaystyle\int_{\mathbb{R}^d} \sigma_1(x-z)\rho(t,z)\,\mathrm{d}z,\ t \geq 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^n, \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v. \end{cases} \tag{2.4}$$

The system is completed by initial data

$$f(0,x,v) = f_0(x,v), \qquad \Psi(0,x,y) = \Psi_0(x,y), \qquad \partial_t\Psi(0,x,y) = \Psi_1(x,y). \tag{2.5}$$

A possible interpretation of the kinetic equation (2.2) consists in considering the model (2.1) for a set of $N \gg 1$ particles. The definition of the self–consistent potential has to be adapted since all the particles interact with the environment, namely we have, for $j \in \{1, ..., N\}$

$$\begin{cases} \ddot{q}_j(t) = -\nabla V(q_j(t)) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q_j(t)-z)\ \sigma_2(y)\ \nabla_x\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \displaystyle\sum_{k=1}^{N} \sigma_1(x-q_k(t)). \end{cases}$$

Note that such a many-particle system is not considered in [17]. It is very likely that its asymptotic behavior is much more complicated than with a single particle because, even if the particles do not interact directly, they do so indirectly via their interaction with the membranes. If we now adopt the mean–field rescaling in which $\Phi \to \frac{1}{N}\Phi$, then (2.2) can be obtained as the limit as N goes to ∞ for the empirical measure $f_N(t,x,v) = \frac{1}{N}\sum_{k=1}^N \delta(x = q_k(t), v = \dot{q}_k(t))$ of the $N-$particle system, assuming the convergence of the initial state $f_N(0,x,v) \to f_0(x,v)$ in some suitable sense. Such a statement can be rephrased in terms of the convergence of the joint distribution of the N–particle system. This issue will be discussed elsewhere [81] and we refer the reader to the lecture notes [43] and to [45] for further information on the mean–field regimes in statistical physics.

In this paper we wish to analyse several aspects of the Vlasov-Wave system (2.2)–(2.5). We warn the reader that, despite the similarities in terminology, the model considered here is very different, both mathematically and physically, from the one dealt with in [15], which is a simplified version of the Vlasov–Maxwell system. It is indeed crucial to understand that the wave equation in this paper is set with variables *transverse* to the physical space: the waves do not propagate at all in the space where the particles move. This leads to very different physical effects; we refer to [17] and references therein for more details on this matter. We add that this paper is far less ambitious than [17], since we do not discuss here the large time behavior of the solutions, only their global existence. As mentioned above, since we dealing with many particles, it is very likely that the question cannot be handled in the same terms as in [17], and that the kinetic model inherits the same technical and conceptual difficulties already mentioned for $N > 1$ particles. We only mention that a particular stationary solution (with f integrable) has been exhibited in [3], and this solution is shown to be linearly stable.

The paper is organized as follows. Section 2.2 contains a preliminary and largely informal discussion to set up notation and to establish some estimates on the interaction potential needed in the bulk of the paper. Section 2.3 establishes the well–posedness of the problem (2.2)–(2.5) (Theorem 2.3.3). We consider a large class of initial data and external potentials with functional arguments which are reminiscent of Dobrushin's analysis of the Vlasov equation [33]. Section 2.4 is devoted to asymptotic issues which allow us to connect (2.2)–(2.5) to Vlasov equations with an *attractive* self–consistent potential. In particular, up to a suitable rescaling of the form function σ_1, we can derive this way the attractive Vlasov–Poisson system. This is quite surprising and unexpected in view of the very different physical motivation of the models.

2.2 Preliminary discussion

Throughout the paper, we make the following assumptions on the model parameters and on the initial conditions. First, on the coupling functions σ_1, σ_2, we impose:

$$\begin{cases} \sigma_1 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}),\ \ \sigma_2 \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), \\ \sigma_1(x) \geq 0,\ \ \sigma_2(y) \geq 0 \text{ for any } x \in \mathbb{R}^d,\ y \in \mathbb{R}^n, \\ \sigma_1, \sigma_2 \text{ are radially symmetric.} \end{cases} \tag{\textbf{H1}}$$

We require that the external potential fulfills

$$\begin{cases} V \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^d), \\ \text{and there exists } C \geq 0 \text{ such that } V(x) \geq -C(1+|x|^2) \text{ for any } x \in \mathbb{R}^d. \end{cases} \tag{\textbf{H2}}$$

This is a rather standard and natural assumption. Note that it ensures global existence when $\sigma_1 = 0 = \sigma_2$: it then implies that the external potential cannot drive the particle to infinity in finite time. For the initial condition of the vibrating environment, we shall assume

$$\Psi_0, \Psi_1 \in L^2(\mathbb{R}^d \times \mathbb{R}^n). \tag{\textbf{H3}}$$

For the initial particle distribution function, we naturally assume

$$f_0 \geq 0, \qquad f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d). \tag{\textbf{H4}}$$

For energy consideration, it is also relevant to suppose

$$\nabla_y \Psi_0 \in L^2(\mathbb{R}^d \times \mathbb{R}^n) \quad \text{ and } \quad \Big((x,v) \mapsto (V(x) + |v|^2) f_0(x,v)\Big) \in L^1(\mathbb{R}^d \times \mathbb{R}^d). \tag{\textbf{H5}}$$

This means that the initial state has finite mass, potential and kinetic energy.

Our goal in this section is to rewrite the equations of the coupled system (2.2)-(2.5) in an equivalent manner, more suitable for our subsequent analysis. The discussion will be informal, with all computations done for sufficiently smooth solutions. The proper functional framework will be provided in the next section. First, we note that it is clear that (2.2) preserves the total mass of the particles

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

In fact, since the field $(v, \nabla_x V + \nabla_x \Phi)$ is divergence–free (with respect to the phase variables (x, v)), any L^p norm of the density f is conserved, $1 \le p \le \infty$. Furthermore, the PDEs system (2.2)–(2.4) inherits from the Hamiltonian nature of the original equations of motion (2.1) the following easily checked energy conservation property:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t\Psi(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y + \frac{c^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_y\Psi(t,x,y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\left(\frac{|v|^2}{2} + V(x) + \Phi(t,x)\right)\mathrm{d}v\,\mathrm{d}x\Bigg\} = 0.\end{aligned}$$

As a matter of fact the energy remains finite when the full set of assumptions (**H1**)–(**H5**) holds.

For the Vlasov–Poisson equation, it is well known that the potential can be expressed by means of a convolution formula. Similarly here, the interaction potential Φ can be computed explicitly as the image of a certain linear operator acting on the macroscopic density $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$; this follows from the fact that the linear wave equation (2.4) can be solved explicitly as the sum of the solution of the homogeneous wave equation with the correct initial conditions plus the retarded solution of the inhomogeneous wave equation. To see how this works, we introduce

$$t \mapsto p(t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\sin(c|\xi|t)}{c|\xi|}\,|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi$$

and

$$\Phi_0(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} \sigma_1(x-z)\left(\widehat{\Psi_0}(z,\xi)\cos(c|\xi|t) + \widehat{\Psi_1}(z,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}\right)\widehat{\sigma_2}(\xi)\,\mathrm{d}z\,\mathrm{d}\xi$$

where the symbol $\widehat{\cdot}$ stands for the Fourier transform with respect to the variable $y \in \mathbb{R}^n$. Note that Φ_0 is the solution of the homogeneous wave equation with the given initial conditions for Ψ. Finally, we define the operator $\mathcal{L}$ which associates to a distribution function $f : (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ the quantity

$$\mathcal{L}(f)(t,x) = \int_0^t p(t-s)\left(\int_{\mathbb{R}^d} \Sigma(x-z)\rho(s,z)\,\mathrm{d}z\right)\mathrm{d}s, \tag{2.6}$$

where

$$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \Sigma = \sigma_1 \underset{x}{*} \sigma_1.$$

We can then check that the pair (f, Ψ) is a solution of (2.2)–(2.4) iff f satisfies

$$\left\{\begin{array}{l}\partial_t f + v\cdot\nabla_x f = \nabla_v f\cdot\nabla_x\left(V + \Phi_0 - \mathcal{L}(f)\right)\\ f(0,x,v) = f_0(x,v)\end{array}\right. \tag{2.7}$$

and Ψ is the unique solution of (2.4).

We sketch the computation, which is instructive. Let (f, Ψ) be a solution of (2.2)–(2.4). Applying the Fourier transform with respect to the variable y we find

$$\left\{\begin{array}{l}(\partial_t^2 + c^2|\xi|^2)\widehat{\Psi}(t,x,\xi) = -(\rho(t,\cdot) \underset{x}{*} \sigma_1)(x)\ \widehat{\sigma_2}(\xi),\\[2ex] \widehat{\Psi}(0,x,\xi) = \widehat{\Psi_0}(x,\xi) \qquad \partial_t\widehat{\Psi}(0,x,\xi) = \widehat{\Psi_1}(x,\xi).\end{array}\right.$$

The solution reads

$$\begin{aligned}\widehat{\Psi}(t,x,\xi) \;\; = \;\; & -\int_0^t (\rho(t-s,\cdot)*\sigma_1)(x)\ \widehat{\sigma_2}(\xi)\frac{\sin(cs|\xi|)}{c|\xi|}\,\mathrm{d}s \\ & +\widehat{\Psi_0}(x,\xi)\cos(c|\xi|t) + \widehat{\Psi_1}(x,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}.\end{aligned} \tag{2.8}$$

To compute Φ in (2.3), we use Plancherel's equality:

$$\begin{aligned}\Phi(t,x) &= \int_{\mathbb{R}^d\times\mathbb{R}^n} \Psi(t,z,y)\sigma_1(x-z)\sigma_2(y)\,\mathrm{d}y\,\mathrm{d}z \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^d\times\mathbb{R}^n} \widehat{\Psi}(t,z,\xi)\sigma_1(x-z)\widehat{\sigma_2}(\xi)\,\mathrm{d}\xi\,\mathrm{d}z \\ &= -\left((\sigma_1*\sigma_1)*\int_0^t\left(\rho(t-s,\cdot)\int_{\mathbb{R}^n}\frac{\sin(cs|\xi|)}{c|\xi|}\frac{|\widehat{\sigma_2}(\xi)|^2}{(2\pi)^n}\,\mathrm{d}\xi\right)\mathrm{d}s\right)(x) \\ &\quad +\frac{1}{(2\pi)^n}\left(\sigma_1*\int_{\mathbb{R}^n}\left(\widehat{\Psi_0}(\cdot,\xi)\cos(c|\xi|t) + \widehat{\Psi_1}(\cdot,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}\right)\ \widehat{\sigma_2}(\xi)\,\mathrm{d}\xi\right)(x) \\ &= -\mathcal{L}(f)(t,x) + \Phi_0(t,x).\end{aligned}$$

Inserting this relation into (2.2), we arrive at (2.7). Conversely, let f be a solution of (2.7) and let Ψ be the unique solution of (2.4). The same computation then shows that Φ in (2.3) is given by $\Phi = \Phi_0 - \mathcal{L}(f)$. Therefore f satisfies (2.2).

The operator $\mathcal{L}$ in (2.6) plays a crucial role in our further analysis. Its precise definition on an appropriate functional space and its basic continuity properties are given in the following Lemma.

Lemma 2.2.1 (Estimates on the interaction potential) *For any $0 < T < \infty$, the following properties hold:*

i) $\mathcal{L}$ *belongs to the space* $\mathcal{A}_T$ *of continuous operators on* $C\Big([0,T];\Big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\Big)'\Big)$ *with values in* $C\Big([0,T];W^{2,\infty}(\mathbb{R}^d)\Big)$*. Its norm is evaluated as follows:*

$$\|\!|\mathcal{L}|\!\|_{\mathcal{A}_T} \leq \|\sigma_1\|^2_{W^{3,2}(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\ \frac{T^2}{2};$$

ii) $\mathcal{L}$ *belongs to the space* $\mathcal{B}_T$ *of continuous operators on* $C\Big([0,T];\Big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\Big)'\Big)$ *with values in* $C^1\Big([0,T];L^{\infty}(\mathbb{R}^d)\Big)$*. Its norm is evaluated as follows:*

$$\|\!|\mathcal{L}|\!\|_{\mathcal{B}_T} \leq \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\ \Big(T+\frac{T^2}{2}\Big);$$

iii) Φ_0 *satisfies*

$$\|\Phi_0(t,\cdot)\|_{W^{2,\infty}(\mathbb{R}^d)} \leq \|\sigma_1\|_{W^{2,2}(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^n)}\ \Big(\|\Psi_0\|_{L^2(\mathbb{R}^n)}+t\|\Psi_1\|_{L^2(\mathbb{R}^n)}\Big)\,,$$

for any $0\leq t\leq T$*, and, moreover*

$$\|\Phi_0\|_{C^1([0,T];L^{\infty}(\mathbb{R}^d))} \leq \|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{W^{1,2}(\mathbb{R}^n)}\ \Big(2\|\Psi_0\|_{L^2(\mathbb{R}^n)}+(1+T)\|\Psi_1\|_{L^2(\mathbb{R}^n)}\Big)\,.$$

Proof. The last statement is a direct consequence of Hölder and Young inequalities; let us detail the proof of items i) and ii). We associate to $f\in\Big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\Big)'$, the macroscopic density $\rho\in\Big(W^{1,\infty}(\mathbb{R}^d)\Big)'$ by the formula:

$$\langle\rho_f,\chi\rangle_{(W^{1,\infty})',W^{1,\infty}(\mathbb{R}^d)} = \langle f,\chi\otimes\mathbf{1}_v\rangle_{(W^{1,\infty})',W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)}\,,\qquad \forall\chi\in W^{1,\infty}(\mathbb{R}^d).$$

Clearly, we have $\|\rho_f\|_{\left(W^{1,\infty}(\mathbb{R}^d)\right)'}\leq\|f\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}$.

For any $\chi\in C^\infty_c(\mathbb{R}^d)$, and $i\in\{0,1,2\}$, we can check the following estimates

$$\begin{aligned}|\langle\rho*\Sigma,\nabla^i\chi\rangle| &= |\langle\rho,(\nabla^i\Sigma)*\chi\rangle|\leq\|\rho\|_{\left(W^{1,\infty}(\mathbb{R}^d)\right)'}\|\,(\nabla^i\Sigma)*\chi\|_{W^{1,\infty}(\mathbb{R}^d)}\\ &\leq \|f\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}\Big(\|\nabla^i\Sigma\|_{L^\infty(\mathbb{R}^d)}+\|\nabla^{i+1}\Sigma\|_{L^\infty(\mathbb{R}^d)}\Big)\,\|\chi\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Since the dual space of L^1 is L^∞, for $i=0$, we deduce that

$$\begin{aligned}\|\rho*\Sigma\|_{L^\infty(\mathbb{R}^d)} &\leq \|f\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}\Big(\|\Sigma\|_{L^\infty(\mathbb{R}^d)}+\|\nabla\Sigma\|_{L^\infty}\Big)\\ &\leq \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)}\|f\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}.\end{aligned}$$

Reasoning similarly for $i=1$ and $i=2$, we obtain

$$\|\rho*\Sigma\|_{W^{2,\infty}(\mathbb{R}^d)}\leq\|\sigma_1\|^2_{W^{3,2}(\mathbb{R}^d)}\|f\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}.$$

We now estimate p. Plancherel's inequality yields

$$|p'(t)| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \cos(c|\xi|t) |\widehat{\sigma_2}(\xi)|^2 \, \mathrm{d}\xi \right| \leq \|\sigma_2\|^2_{L^2(\mathbb{R}^n)}.$$

Since $p(0) = 0$, it follows that $|p(t)| \leq \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} t$. Hence, for all $0 \leq t \leq T < \infty$, we have

$$\begin{aligned} \|\mathcal{L}(f)(t)\|_{W^{2,\infty}(\mathbb{R}^d \times \mathbb{R}^d)} &\leq \|\Sigma * \rho\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d))} \int_0^t |p(t-s)| \, \mathrm{d}s \\ &\leq \|f\|_{C\left([0,T];\left(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\right)'\right)} \|\sigma_1\|^2_{W^{3,2}(\mathbb{R}^d)} \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} \frac{T^2}{2}. \end{aligned}$$

This proves the estimate in i). That $\mathcal{L}(f)(t)$ is continuous as a function of t follows easily from the previous argument. As a further by-product note that

$$\|\mathcal{L}(f)(t)\|_{L^\infty} \leq \|f\|_{C\left([0,T];\left(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\right)'\right)} \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)} \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} \frac{T^2}{2}$$

holds. Since $p(0) = 0$, we have

$$\partial_t \mathcal{L}(f)(t) = \int_0^t p'(t-s) \Sigma * \rho(s) \, \mathrm{d}s$$

which gives:

$$\|\partial_t \mathcal{L}(f)(t)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq \|f\|_{C\left([0,T];\left(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\right)'\right)} \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)} \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} T.$$

This ends the proof of ii). ■

2.3 Existence of solutions

The proof of existence of solutions to (2.7) relies on estimates satisfied by the characteristics curves defined by the following ODE system:

$$\left\{ \begin{array}{l} \dot{X}(t) = \xi(t), \\ \dot{\xi}(t) = -\nabla V(X(t)) - \nabla \Phi(t, X(t)). \end{array} \right. \tag{2.9}$$

From now on, we adopt the following notation. The potential Φ being given, we denote by $\varphi_\alpha^{\Phi,t}(x_0, v_0) \in \mathbb{R}^d \times \mathbb{R}^d$ the solution of (2.9) which starts from (x_0, v_0) at time $t = \alpha$: the initial data is $\varphi_\alpha^{\Phi,\alpha}(x_0, v_0) = (x_0, v_0)$. We use the shorthand notation $t \mapsto (X(t), \xi(t))$ for $t \mapsto \varphi_0^{\Phi,t}(x_0, v_0)$, the solution of (2.9) with $X(0) = x_0$ and $\xi(0) = v_0$. Owing to the regularity of V, $\mathcal{L}$ and Φ_0, see Lemma 2.2.1, the solution of the differential system (2.9) is indeed well defined for prescribed initial data; it also allows us to establish the following estimates, where characteristics are evaluated both forward and backward.

Lemma 2.3.1 (Estimates on the characteristic curves) *Let V satisfy* (**H2**) *and let* $\Phi \in C^0([0, \infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0, \infty); L^\infty(\mathbb{R}^d))$.

a) There exists a function $(\mathscr{N}, t, x, v) \in [0,\infty) \times [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \mapsto R(\mathscr{N}, t, x, v) \in [0,\infty)$*, non decreasing with respect to the first two variables, such that the solution* $t \mapsto (X(t), \xi(t))$ *of* (2.9) *with initial data* $X(0) = x_0$, $\xi(0) = v_0$*, satisfies the following estimate, for any* $t \in \mathbb{R}$*,*

$$(X(t), \xi(t)) \in B\left(0, R\left(\|\Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}, |t|, x_0, v_0\right)\right) \subset \mathbb{R}^d \times \mathbb{R}^d.$$

b) Taking two different additional potential Φ_1 *and* Φ_2*, the following two estimates hold for any* $t > 0$*:*

$$\begin{aligned}
&|(\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t})(x_0, v_0)| \\
&\quad \leq \int_0^t \|(\Phi_1 - \Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)} \exp\left(\int_s^t \|\nabla^2(\Phi_1(\tau) + V)\|_{L^\infty(B_\tau(x_0,v_0))} \,\mathrm{d}\tau\right) \mathrm{d}s, \\
&|(\varphi_t^{\Phi_1,0} - \varphi_t^{\Phi_2,0})(x, v)| \\
&\quad \leq \int_0^t \|(\Phi_1 - \Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)} \exp\left(\int_0^s \|\nabla^2(\Phi_1(\tau) + V)\|_{L^\infty(\tilde{B}_{t,\tau}(x,v))} \,\mathrm{d}\tau\right) \mathrm{d}s,
\end{aligned}$$

where we set $B_\tau(x,v) = B\left(0, R\left(\max_{i=1,2} \|\Phi_i\|_{C^1([0,\tau];L^\infty(\mathbb{R}^d))}, \tau, x, v\right)\right)$ *in the first inequality and* $\tilde{B}_{t,\tau} = B\left(0, R\left(\max_{i=1,2} \|\Phi_i\|_{C^1([\tau,t];L^\infty(\mathbb{R}^d))}, t-\tau, x, v\right)\right)$ *in the second one.*

The proof of the lemma is postponed the end of this section. Given $0 < R_0 < \infty$, and Ψ_0, Ψ_1 satisfying (**H3**) (they enter into the definition of Φ_0), we set

$$r(t, x, v) = R(\|\Phi_0\|_{C^1([0,t];L^\infty(\mathbb{R}^d))} + \|\!|\mathcal{L}|\!\|_{\mathcal{B}_t} R_0, t, x, v). \tag{2.10}$$

Where we remind the reader that

$$\|\!|\mathcal{L}|\!\|_{\mathcal{B}_t} = \sup_{f \neq 0} \frac{\|\mathcal{L}(f)\|_{C^1(0,T;L^\infty(\mathbb{R}^d))}}{\|f\|_{C(0,T;(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d))')}}$$

is estimated in the lemma 2.2.1. Proving uniqueness statements for the wide class of external potentials considered in(**H2**) requires to strengthen the hypothesis on the initial data.

Definition 2.3.2 *Let* $0 < T, R_0 < \infty$*. We say that an integrable function* f_0 *belongs to the set* $E_{R_0,T}$ *if* $f_0 \geq 0$ *satisfies* $\|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \leq R_0$ *and, furthermore,*

$$\mathscr{K}_{R_0,T}(f_0) := \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x, v) \exp\left(\int_0^T \|\nabla^2 V\|_{L^\infty(B(0,r(t,x,v)))} \,\mathrm{d}t\right) \mathrm{d}v \,\mathrm{d}x < \infty.$$

Theorem 2.3.3 *Assume* (**H1**)–(**H3**)*. Let* $0 < R_0, T < \infty$*. Let* $f_0 \in E_{R_0,T}$*. Then, there exists a unique* $f \in C([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *weak solution of* (2.7)*. The solution is continuous with respect to the parameters* $\mathcal{L}$, Φ_0 *and* f_0*, respectively in* $\mathcal{A}_T \cap \mathcal{B}_T$, $C^1([0,\infty); W^{2,\infty}(\mathbb{R}^d))$ *and* $E_{R_0,T}$*. If* $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$ *only, see* (**H4**)*, then there exists* $f \in C([0,\infty); L^1(\mathbb{R}^d \times \mathbb{R}^d))$*, weak solution of* (2.7)*.*

The statement can be rephrased for the original problem (2.2)–(2.5). We also establish the conservation of energy.

Corollary 2.3.4 *Assume* (**H1**)–(**H3**). *Let* $0 < R_0, T < \infty$. *Let* $f_0 \in E_{R_0,T}$. *Then, there exists a unique weak solution* (f, Ψ) *to the system* (2.2)–(2.5) *with* $f \in C([0,T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *and* $\Psi \in C([0,T]; L^2(\mathbb{R}^d \times \mathbb{R}^n))$. *The solution is continuous with respect to the parameters* σ_1, σ_2, Ψ_0, Ψ_1 *and* f_0 *in the sets* $W^{3,2}(\mathbb{R}^d)$, $L^2(\mathbb{R}^n)$, $L^2(\mathbb{R}^d \times \mathbb{R}^n)$, $L^2(\mathbb{R}^d \times \mathbb{R}^n)$ *and* $E_{R_0,T}$, *respectively. If* f_0 *satisfies* (**H4**) *only, then there exists a weak solution with* $f \in C([0,\infty); L^1(\mathbb{R}^d \times \mathbb{R}^d))$ *and* $\Psi \in C([0,T]; L^2(\mathbb{R}^d \times \mathbb{R}^n))$. *Furthermore, when the initial data satisfies* (**H5**) *the total energy*

$$\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t \Psi(t,x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y + \frac{c^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_y \Psi(t,x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y \\ + \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\left(\frac{|v|^2}{2} + V(x) + \Phi(t,x)\right)\mathrm{d}v\,\mathrm{d}x$$

is conserved.

Remark 2.3.5 *Definition 2.3.2 restricts the set of initial data depending on the growth of the Hessian of the external potential. Of course, any integrable data* f_0 *with compact support fulfils the criterion in Definition 2.3.2, and when the potential has at most quadratic growth, any data satisfying* (**H4**) *is admissible. As it will be clear within the proof, the continuity with respect to the initial data does not involve the* L^1 *norm only, but the more intricate quantity* $\mathscr{K}_{R_0,T}$ *also arises in the analysis.*

Remark 2.3.6 *The present approach does not need a restriction on the transverse dimension* ($n \geq 3$ *in [17]). The proof can be slightly modified to treat the case of measure–valued initial data* f_0, *thus including the results in [17] for a single particle* ($f_0(x,v) = \delta_{(x=x_0,v=v_0)}$), *and we can consider a set of* $N > 1$ *particles as well. The measure–valued solution is then continuous with respect to the initial data in* $C([0,T]; (W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d))')$. *This viewpoint will be further detailed with the discussion of mean–field asymptotics [81].*

The proof of Theorem 2.3.3 relies on a fixed point strategy, the difficulty being to set up the appropriate functional framework. It turns out that it will be convenient to work with the $C\Big([0,T]; (W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d))'\Big)$ norm. We remind the reader that the dual norm on $(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d))'$ is equivalent to the Kantorowich–Rubinstein distance

$$W_1(f,g) = \sup_{\pi}\left\{\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |\zeta - \zeta'| \,\mathrm{d}\pi(\zeta,\zeta')\right\}$$

where the supremum is taken over measures π having f and g as marginals, see e. g. [82, Remark 6.5]. This distance appears naturally in the analysis of Vlasov–like systems, as pointed out in [33]. In order to define the fixed point procedure, we introduce the following mapping. For a non negative integrable function f_0, we denote by Λ_{f_0} the application which associates to Φ in $C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$ the unique solution f of the Liouville equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_v f\cdot\nabla_x (V+\Phi) = 0,$$

with initial data f_0. We shall make use of the following statement, which provides useful estimates.

Lemma 2.3.7 *For any $f_0\in L^1(\mathbb{R}^d\times\mathbb{R}^d)$, the application Λ_{f_0} is continuous on the set $C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$ with values in $C([0,\infty);L^1(\mathbb{R}^d\times\mathbb{R}^d))$. Furthermore, we have*

$$\|\Lambda_{f_0}(\Phi)-\Lambda_{g_0}(\Phi)\|_{L^\infty(0,\infty;L^1(\mathbb{R}^d\times\mathbb{R}^d))} = \|f_0-g_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)},$$

for any $\Phi\in C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$.

Proof. Let $0<T<\infty$ be fixed once for all. We begin by assuming that f_0 is C^1 and compactly supported. For any $0\le t\le T$, we have

$$\Lambda_{f_0}(\Phi)(t) = f_0\circ\varphi_t^{\Phi,0},$$

where we remind the reader that $\varphi_t^{\Phi,0}(x,v)$ stands for the evaluation at time 0 of the solution of (2.9) which starts at time t from the state (x,v). Accordingly any L^p norm is preserved: $\|\Lambda_{f_0}(\Phi)(t)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}$ holds for any $t\ge 0$ and any $1\le p\le\infty$. By linearity, this immediately proves the continuity estimate with respect to the initial data.
To establish the continuity properties with respect to Φ, we first observe, denoting $\Lambda_{f_0}(\Phi)=f$, that $(x,v)\in\operatorname{supp}(f(t,\cdot))$ iff $\varphi_t^{\Phi,0}(x,v)\in\operatorname{supp}(f_0)$, that is $(x,v)\in\varphi_0^{\Phi,t}(\operatorname{supp}(f_0))$. Therefore, by Lemma 2.3.1, we can find a compact set $K_T\subset\mathbb{R}^d\times\mathbb{R}^d$ such that $\operatorname{supp}(f(t,\cdot))\subset K_T$ for any $0\le t\le T$. We are dealing with potentials Φ_1 and Φ_2 in $C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$. We can again find a compact set, still denoted by $K_T\subset\mathbb{R}^d\times\mathbb{R}^d$, such that the support of the associated solutions $\Lambda_{f_0}(\Phi_1)$ and $\Lambda_{f_0}(\Phi_2)$ for any $0\le t\le T$ is contained in K_T. We infer that

$$\|\Lambda_{f_0}(\Phi_1)(t)-\Lambda_{f_0}(\Phi_2)(t)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \int_{K_T}|f_0\circ\varphi_t^{\Phi_1,0}-f_0\circ\varphi_t^{\Phi_2,0}|\,\mathrm{d}v\,\mathrm{d}x$$

$$\le \|f_0\|_{W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)}\ \operatorname{meas}(K_T)\ \sup_{(x,v)\in K_T}|\varphi_t^{\Phi_1,0}(x,v)-\varphi_t^{\Phi_2,0}(x,v)|$$

holds. As τ ranges over $[0,t]\subset[0,T]$ and (x,v) lies in K_T, the backward characteristics $\varphi_t^{\Phi_i,\tau}(x,v)$ still belong to a compact set. We introduce the following quantities

$$\mathcal{R} = \sup_{(x,v)\in K_T} R\left(\max_{i=1,2}\|\Phi_i\|_{C^1([0,T];L^\infty(\mathbb{R}^d))},T,x,v\right)$$

and

$$m_T = \exp\left(\int_0^T \|\nabla^2\Phi_1(u)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u\right).$$

For $0 \le t \le T$ and any $(x,v) \in K_T$, Lemma 2.3.1-b) yields:

$$\begin{aligned}
&|\varphi_t^{\Phi_1,0}(x,v) - \varphi_t^{\Phi_2,0}(x,v)| \\
&\qquad\qquad \le m_T \int_0^t \|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)} \exp\left(\int_0^s \|\nabla^2 V\|_{L^\infty(B(0,\mathcal{R}))}\,\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}$$

We conclude with

$$\sup_{(x,v)\in K_T} |\varphi_t^{\Phi_1,0}(x,v) - \varphi_t^{\Phi_2,0}(x,v)| \xrightarrow[\substack{\|\Phi_1-\Phi_2\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d))}\to 0\\ \|\Phi_1\|_{C^1([0,T];L^\infty(\mathbb{R}^d))},\|\Phi_2\|_{C^1([0,T];L^\infty(\mathbb{R}^d))}\le M}]{} 0.$$

(It is important to keep both the $C^1([0,T];L^\infty(\mathbb{R}^d))$ and $L^\infty(0,T;W^{2,\infty}(\mathbb{R}^d))$ norms of the potentials bounded since these quantities appear in the definition of $\mathcal{R}$ and m_T.) It proves the asserted continuity of the solution with respect to the potential. By uniform continuity of the flow on the compact set $[0,T]\times K_T$, we obtain the time continuity. Hence the result is proved when the initial data f_0 lies in C_c^1.
We finally extend the result for initial data f_0 in L^1. Those can be approximated by a sequence $\left(f_0^k\right)_{k\in\mathbb{N}}$ of functions in $C_c^1(\mathbb{R}^d\times\mathbb{R}^d)$. We have

$$\|\Lambda_{f_0}(\Phi)(t) - \Lambda_{f_0^k}(\Phi)(t)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \|\Lambda_{(f_0-f_0^k)}(\Phi)(t)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_0 - f_0^k\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Therefore, Λ_{f_0} is the uniform limit of maps which are continuous with respect to Φ and the time variable. This remark ends the proof. ■

Proof of Theorem 2.3.3.
Existence–uniqueness for initial data in $E_{R_0,T}$.
We turn to the fixed point reasoning. For f given in $C\Big([0,T];(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d))'\Big)$, we set

$$\mathcal{T}_{f_0}(f) = \Lambda_{f_0}(\Phi_0 - \mathcal{L}(f)).$$

It is clear that a fixed point of $\mathcal{T}_{f_0}$ is a solution to (2.7). Note also that, as a consequence of Lemma 2.2.1 and Lemma 2.3.7, $\mathcal{T}_{f_0}(f)(t) \in L^1(\mathbb{R}^d\times\mathbb{R}^d)$. More precisely, we know that $f \mapsto \mathcal{T}(f)$ is continuous with values in the space $C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d)) \subset C\Big([0,T];(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d))'\Big)$. We shall prove that $\mathcal{T}$ admits an iteration which is a contraction on the ball with centre 0 and radius R_0.
Let f_1 and f_2 be two elements of this ball. We denote $\varphi_\alpha^{\Phi_i,t}$ the flow of (2.9) with $\Phi_i = \Phi_0 - \mathcal{L}(f_i)$: $\varphi_\alpha^{\Phi_i,t}(x_0,v_0)$ satisfies (2.9) with (x_0,v_0) as data at time $t=\alpha$. Let χ be a trial function in $W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} (\mathcal{T}(f_1)(t,x,v) - \mathcal{T}(f_2)(t,x,v))\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\right| \\
&\quad = \left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \left(f_0\circ\varphi_t^{\Phi_1,0} - f_0\circ\varphi_t^{\Phi_2,0}\right)(x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\right| \\
&\quad = \left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\left(\chi\circ\varphi_0^{\Phi_1,t} - \chi\circ\varphi_0^{\Phi_2,t}\right)(x,v)\,\mathrm{d}v\,\mathrm{d}x\right| \\
&\quad \le \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\|\nabla\chi\|_\infty \left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right|(x,v)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

It follows that

$$\|\mathcal{T}(f_1)(t)-\mathcal{T}(f_2)(t)\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'} \le \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\left|\varphi_0^{\Phi_1,t}-\varphi_0^{\Phi_2,t}\right|(x,v)\,\mathrm{d}v\,\mathrm{d}x. \qquad (2.11)$$

By using Lemma 2.3.1-b), we obtain

$$\begin{aligned}&\left|\varphi_0^{\Phi_1,t}-\varphi_0^{\Phi_2,t}\right|(x,v)\\ &\quad\le \bar m_T\int_0^t \|\mathcal{L}(f_1-f_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))}\\ &\qquad\qquad\times\exp\left(\int_s^t \|\nabla^2 V\|_{L^\infty B(0,R(\|\Phi_0+\mathcal{L}(f_i)\|_{C^1([0,u];L^\infty(\mathbb{R}^d))},u,x_0,v_0))}\,\mathrm{d}u\right)\mathrm{d}s,\end{aligned}$$

where we have used

$$\begin{aligned}&\exp\left(\int_0^T \|\nabla^2(\Phi_0(u)-\mathcal{L}(f_1)(u)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u\right)\\ &\qquad\le \exp\left(\int_0^T\left(\|\nabla^2\Phi_0(u)\|_{L^\infty(\mathbb{R}^d)}+\|\!|\mathcal{L}|\!\|_{\mathcal{A}_u}\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\right)\mathrm{d}u\right)=\bar m_T.\end{aligned}$$

Plugging this estimate into (2.11) yields

$$\begin{aligned}&\|\mathcal{T}(f_1)(t)-\mathcal{T}(f_2)(t)\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}\\ &\quad\le \bar m_T\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\int_0^t\|\mathcal{L}(f_1-f_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))}\\ &\qquad\times\exp\left(\int_s^t\|\nabla^2 V\|_{L^\infty(B(0,r(u,x,v)))}\,\mathrm{d}u\right)\mathrm{d}s\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

It recasts as

$$\|\mathcal{T}(f_1)(t)-\mathcal{T}(f_2)(t)\|_{(W^{1,\infty})'}\le \bar m_T'\mathscr{K}_{R_0,T}\int_0^t\|f_1-f_2\|_{L^\infty\left(0,s;\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'\right)}\,\mathrm{d}s$$

with

$$\bar m_T'=\bar m_T\times\sup_{0\le s\le T}\|\!|\mathcal{L}|\!\|_{\mathcal{A}_s}.$$

By induction, we deduce that

$$\|\mathcal{T}^\ell(f_1)(t)-\mathcal{T}^\ell(f_2)(t)\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}\le\frac{\left(t\bar m_T'\mathscr{K}_{R_0,T}\right)^\ell}{\ell!}\|f_1-f_2\|_{L^\infty\left(0,T;\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'\right)}$$

holds for any $\ell\in\mathbb{N}$ and $0\le t\le T$. Finally, we are led to

$$\|\mathcal{T}^\ell(f_1)-\mathcal{T}^\ell(f_2)\|_{L^\infty\left(0,T;\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'\right)}\le\frac{\left(T\bar m_T'\mathscr{K}_{R_0,T}\right)^\ell}{\ell!}\|f_1-f_2\|_{L^\infty\left(0,T;\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'\right)}.$$

It shows that an iteration of $\mathcal{T}$ is a contraction. Therefore, there exists a unique fixed point f in $C\Big([0,T];(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d))'\Big)$. Furthermore, $f=\mathcal{T}(f)\in C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d))$, and the solution is continuous with respect to the parameters of the system. Note that the continuity estimate involves the quantity in Definition 2.3.2 which restricts the growth assumption of the initial data.

Step 2: Existence for an integrable data

We proceed by approximation. Let f_0 be in $L^1(\mathbb{R}^d\times\mathbb{R}^d)$, with $\|f_0\|_{L^1}\leq R_0$. Then,

$$(x,v)\mapsto f_0^k(x,v)=f_0(x,v)\mathbf{1}_{\sqrt{x^2+v^2}\leq k}$$

lies in $E_{R_0,T}$ (with a constant $\mathscr{K}_{R_0,T}$ which can blow up as $k\to\infty$). The previous step defines f^k, solution of (2.7) with this initial data. Of course we wish to conclude by passing to the limit $k\to\infty$. However, the necessary compactness arguments are not direct and the proof splits into several steps.

We start by showing that the sequence $\left(f^k\right)_{k\in\mathbb{N}}$ is compact in $C([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{weak}-\star)$. Pick $\chi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. For any $0\leq t\leq T$, we have, on the one hand,

$$\begin{aligned}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\right| &\leq \|f^k(t,\cdot)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\leq \|f_0^k\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\leq \|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)},\end{aligned}\tag{2.12}$$

and, on the other hand,

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\right|\\ &\quad=\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\,\Big(v\cdot\nabla_x\chi-\nabla_x(V+\Phi_0-\mathcal{L}(f)(t))\cdot\nabla_v\chi\Big)(x,v)\,\mathrm{d}v\,\mathrm{d}x\right|\\ &\quad\leq\|f_0\|_{L^1}\Big(\|v\cdot\nabla_x\chi-\nabla V\cdot\nabla_v\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d))}\\ &\qquad\qquad+\Big(|\!|\!|\mathcal{L}|\!|\!|_{\mathcal{A}_T}\|f_0\|_{L^1}+\|\Phi_0\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}\Big)\|\nabla_v\chi\|_{L^\infty}\Big).\end{aligned}$$

Lemma 2.2.1 then ensures that the set

$$\left\{t\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x,\ k\in\mathbb{N}\right\}$$

is equibounded and equicontinuous; hence, by virtue of Arzela–Ascoli's theorem it is relatively compact in $C([0,T])$. Going back to (2.12), a simple approximation argument allows us to extend the conclusion to any trial function χ in $C_0(\mathbb{R}^d\times\mathbb{R}^d)$, the space of continuous functions that vanish at infinity.

This space is separable; consequently, by a diagonal argument, we can extract a subsequence and find a measure valued function $t\mapsto\ \mathrm{d}f(t)\in\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$ such that

$$\lim_{k\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x=\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}f(t)$$

holds uniformly on $[0,T]$, for any $\chi\in C_0(\mathbb{R}^d\times\mathbb{R}^d)$. As a matter of fact, we note that $\mathrm{d}f$ is non negative and for any $0\leq t\leq T$ it satisfies

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}f(t)\leq\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Next, we establish the tightness of the sequence of approximate solutions. Let $\epsilon>0$ be fixed once for all. We can find $M_\epsilon>0$ such that

$$\int_{x^2+v^2\geq M_\epsilon^2} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x\leq\epsilon.$$

Let us set

$$A_\epsilon = \sup\{r(T,x,v),\ (x,v) \in B(0,M_\epsilon)\}$$

where we remind the reader that $r(T,x,v)$ has been defined in (2.10): $0 < A_\epsilon < \infty$ is well defined by Lemma 2.2.1. Let $\varphi_\alpha^{k,t}$ stand for the flow associated to the characteristics of the equation satisfied by f^k. For any $0 \le t \le T$, setting $\complement U$ the complement of U in $\mathbb{R}^d \times \mathbb{R}^d$, we have $\varphi_0^{k,t}(B(0,M_\epsilon)) \subset B(0,A_\epsilon)$ so that $\complement\left(\varphi_t^{k,0}(B(0,A_\epsilon))\right) = \varphi_t^{k,0}\Big(\complement B(0,A_\epsilon)\Big) \subset \complement B(0,M_\epsilon)$. It follows that

$$\begin{aligned}\int_{\complement B(0,A_\epsilon)} f^k(t,x,v)\,\mathrm{d}v\,\mathrm{d}x &= \int_{\complement B(0,A_\epsilon)} f_0^k(\varphi_t^{k,0}(x,v))\,\mathrm{d}v\,\mathrm{d}x \\ &= \int_{\complement\varphi_t^{k,0}(B(0,A_\epsilon))} f_0^k(x,v)\,\mathrm{d}v\,\mathrm{d}x \\ &\le \int_{\complement B(0,M_\epsilon)} f_0(x,v)\,\mathrm{d}v\,\mathrm{d}x \le \epsilon.\end{aligned}$$

By a standard approximation, we check that the same estimate is satisfied by the limit f:

$$\int_{\complement B(0,A_\epsilon)} \mathrm{d}f(t) \le \epsilon.$$

Finally, we justify that f^k converges to f in $C([0,T];(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d))')$. Pick χ in $W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, with $\|\chi\|_{W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)} \le 1$. We introduce a cut-off function θ_R as follows:

$$\begin{array}{ll}\theta_R(x,v) = \theta(x/R, v/R), & \theta \in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d),\\ \theta(x,v) = 1 \text{ for } \sqrt{x^2+v^2} \le 1, & \theta(x) = 0 \text{ for } x^2+v^2 \ge 4,\\ 0 \le \theta(x) \le 1 \text{ for any } x \in \mathbb{R}^d. & \end{array} \tag{2.13}$$

Then, we split

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}f(t)\\ &\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi\theta_R(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\theta_R(x,v)\,\mathrm{d}f(t)\\ &\qquad+ \int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(1-\theta_R)(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(1-\theta_R)(x,v)\,\mathrm{d}f(t).\end{aligned}$$

Choosing $R \ge A_\epsilon$ yields

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(1-\theta_R)(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(1-\theta_R)(x,v)\,\mathrm{d}f(t)\Big|\\ &\qquad\qquad\le 2\epsilon\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.\end{aligned} \tag{2.14}$$

By virtue of the Arzela-Ascoli theorem, $W^{1,\infty}(B(0,2R))$ embeds compactly in $C(B(0,2R))$. Thus, we can find a family $\{\chi_1,...,\chi_{m_\epsilon}\}$ of functions in $W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ such that, for any $\chi \in W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, $\|\chi\|_{W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)} \le 1$, there exists an index $i \in \{1,...,m_\epsilon\}$ with $\|\theta_R\chi - \chi_i\|_{L^\infty(B(0,2R))} \le \epsilon$ (since $\chi\theta_R$ lies in a bounded ball of $W^{1,\infty}(B(0,2R))$). Therefore, let us write

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi\theta_R(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\theta_R(x,v)\,\mathrm{d}f(t)\\
&\quad= \int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi_i(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_i(x,v)\,\mathrm{d}f(t)x\\
&\qquad + \int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)(\chi\theta_R-\chi_i)(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}(\chi\theta_R-\chi_i)(x,v)\,\mathrm{d}f(t),
\end{aligned}$$

where the last two terms can both be dominated by $\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\epsilon$. We thus arrive at

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}f(t)\right|\\
&\quad\le 2\epsilon(\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)})\\
&\qquad + \left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\ \chi_i(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_i(x,v)\ \ \mathrm{d}f(t)\right|\\
&\quad\le 2\epsilon(\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} + \|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)})\\
&\qquad + \sup_{j\in\{1,\dots,m_\epsilon\}}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\ \chi_i(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_i(x,v)\ \ \mathrm{d}f(t)\right|,
\end{aligned}$$

for any $\chi\in W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, with $\|\chi\|_{W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)}\le 1$. The last term can be made smaller than ϵ by choosing $k\ge N_\epsilon$ large enough. In other words, we can find $N_\epsilon\in\mathbb{N}$ such that

$$\begin{aligned}
\sup_{\|\chi\|_{W^{1,\infty}}\le 1}&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f^k(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}f(t)\right|\\
&\le 2\epsilon(2+\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)})
\end{aligned}$$

holds for any $0\le t\le T$, and $k\ge N_\epsilon$: f^k converges to f in $C\Big([0,T];(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d))'\Big)$. According to Lemma 2.3.7, together with Lemma 2.2.1, it implies that $\mathcal{T}_{f_0}(f^k)$ converges to $\mathcal{T}_{f_0}(f)$ in $C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d))$.

By definition $\mathcal{T}_{f_0^k}(f^k)=f^k$ so that

$$\begin{aligned}
&\|f^k-\mathcal{T}_{f_0}(f)\|_{C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d))}\\
&\quad\le \|\mathcal{T}_{f_0^k}(f^k)-\mathcal{T}_{f_0}(f^k)\|_{C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d)))} + \|\mathcal{T}_{f_0}(f^k)-\mathcal{T}_{f_0}(f)\|_{C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d)))}\\
&\quad\le \|f_0^k-f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} + \|\mathcal{T}_{f_0}(f^k)-\mathcal{T}_{f_0}(f)\|_{C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d)))}
\end{aligned}$$

holds, where we have used Lemma 2.3.7 again. Letting k go to ∞, we realize that f^k also converges to $\mathcal{T}_{f_0}(f)$ in $C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d))$. It implies both $f=\mathcal{T}_{f_0}(f)$ and $f\in C([0,T];L^1(\mathbb{R}^d\times\mathbb{R}^d))$. By definition of $\mathcal{T}_{f_0}$, f satisfies (2.7), and it also justifies that f is absolutely continuous with respect to the Lebesgue measure, which ends the proof. ∎

Proof of Lemma 2.3.1. Let (X,ξ) be the solution of (2.9) with $(X(0),\xi(0))=(x_0,v_0)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V(X(t))+\Phi(t,X(t))+\frac{|\xi(t)|^2}{2}\right]=(\partial_t\Phi)(t,X(t)).$$

The right hand side is dominated by $\|\partial_t \Phi\|_{C([0,t];L^\infty(\mathbb{R}^d))}$. With $t \geq 0$, integrating this relation yields

$$\frac{|\xi(t)|^2}{2} \leq \left(V(x_0) + \Phi(0,x_0) + \frac{|v_0|^2}{2}\right) - (V(X(t)) + \Phi(t,X(t))) + t\|\partial_t \Phi\|_{C([0,t];L^\infty(\mathbb{R}^d))}.$$

Owing to (**H2**) we deduce that

$$|\xi(t)|^2 \leq a(t) + 2C|X(t)|^2$$

holds with

$$a(t) = 2\left|V(x_0) + \Phi(0,x_0) + \frac{|v_0|^2}{2}\right| + 2t\|\partial_t \Phi\|_{C([0,t];L^\infty(\mathbb{R}^d))} + 2\|\Phi(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + 2C.$$

Next, we simply write

$$\frac{\mathrm{d}|X(t)|^2}{\mathrm{d}t}(t) = 2X(t)\cdot\xi(t) \leq X(t)^2 + \xi(t)^2$$

so that the estimate just obtained on ξ yields

$$|X(t)|^2 \leq |x_0| + (1+2C)\int_0^t |X(s)|^2\,\mathrm{d}s + \int_0^t a(s)\,\mathrm{d}s.$$

By using the Grönwall lemma we conclude that

$$|X(t)|^2 \leq |x_0|^2 e^{(1+2C)t} + \int_0^t e^{(1+2C)(t-s)} a(s)\,\mathrm{d}s$$

holds. Going back to the velocity, we obtain

$$|\xi(t)|^2 \leq 2C\left(|x_0|^2 e^{(1+2C)t} + \int_0^t e^{(1+2C)(t-s)} a(s)\,\mathrm{d}s\right) + a(t).$$

In order to simplify, we set $\alpha = 1+2C$, we get the estimates

$$|X(t)|^2 + |\xi(t)|^2 \leq \alpha\left(|x_0|^2 e^{\alpha t} + \int_0^t e^{\alpha(t-s)} a(s)\,\mathrm{d}s\right) + a(t).$$

For all $0 \leq s \leq t$, in order to estimate the right hand side, we use the obvious control

$$\|\partial_t \Phi\|_{C([0,s];L^\infty(\mathbb{R}^d))}, \|\Phi(s,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|\partial_t \Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))} =: \mathscr{N}.$$

After some simplification, we finally obtain the result for

$$R(\mathscr{N},t,x_0,v_0) = \sqrt{2}\left(\frac{\alpha}{2}|x_0|^2 + |V(x_0)| + \frac{|v_0|^2}{2} + \frac{\mathscr{N}}{\alpha} + 2\mathscr{N} + C\right)^{1/2} e^{\alpha t/2}.$$

It concludes the proof of Lemma 2.3.1-a).

Next, let (X_1,ξ_1) and (X_2,ξ_2) be two solutions of (2.9) with the same initial data (x_0,v_0), but different potentials Φ_1,Φ_2. We already know that the two characteristic curves $(X_i(s),\xi_i(s))$, for $i \in \{1,2\}$, belong to $B_s(x,v)$. We have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}|X_1(s)-X_2(s)| \le |\xi_1(s)-\xi_2(s)|, \\ \dfrac{\mathrm{d}}{\mathrm{d}s}|\xi_1(s)-\xi_2(s)| \le \|\nabla\left(\Phi_1(s,\cdot)-\Phi_2(s,\cdot)\right)\|_{L^\infty(\mathbb{R}^d)} \\ \qquad\qquad\qquad\qquad +|X_1(s)-X_2(s)|\|\nabla^2(V+\Phi_1(s,\cdot))\|_{L^\infty(B_s(x,v))} \end{cases}$$

The Grönwall lemma yields the estimate

$$\begin{aligned} &|(X_1(t),\xi_1(t))-(X_2(t),\xi_2(t))| \\ &\le \int_0^t \|(\Phi_1-\Phi_2)(\tau,\cdot)\|_{W^{1,\infty}(\mathbb{R}^d)} \exp\left(\int_s^t \left(\|\nabla^2(V+\Phi_1(u))\|_{L^\infty(B_u(x,v))}\right)\mathrm{d}u\right)\mathrm{d}s. \end{aligned}$$

Finally, we wish to evaluate the backward characteristics, looking at the state at time 0, given the position/velocity pair at time t. Namely we consider $\varphi_t^{\Phi,s}(x,v)$ for $s \le t$, bearing in mind $\varphi_t^{\Phi,t}(x,v)=(x,v)$. We set

$$\begin{pmatrix} Y \\ \zeta \end{pmatrix}(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \varphi_t^{\Phi,t-s}(x,v).$$

We check that (Y,ζ) satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}Y(s)=\zeta(s), & \dfrac{\mathrm{d}}{\mathrm{d}s}\zeta(s) = -\nabla V(Y(s)) - \nabla\Phi(t-s,Y(s)), \\ Y(0)=x, & \zeta(0)=v. \end{cases}$$

Changing Φ for $\Phi(t-\cdot)$, this allows us to obtain the same estimates on (Y,ζ) for all $s \ge 0$. We conclude by taking $s=t$. ■

Proof of Corollary 2.3.4. Theorem 2.3.3 constructs solutions to (2.7) in $C^0([0,\infty);L^1(\mathbb{R}^d\times\mathbb{R}^d))$. We have now the functional framework necessary to justify the manipulations made in Section 2.2. For Ψ_0,Ψ_1 verifying (**H3**), formula (2.8) defines a solution $\Psi \in C([0,\infty);L^2(\mathbb{R}^n\times\mathbb{R}^d))$ of the wave equation, and finally (f,Ψ) satisfies (2.2)–(2.5). Conversely, if $f \in C^0([0,\infty);L^1(\mathbb{R}^d\times\mathbb{R}^d))$ and $\Psi \in C([0,\infty);L^2(\mathbb{R}^n\times\mathbb{R}^d))$ is a solution of the system (2.2)–(2.5), then we can rewrite $\Phi = \Phi_0 - \mathcal{L}(f)$ and f verifies (2.7). This equivalence justifies the first part of the statement in Corollary 2.3.4.

It only remains to justify the energy conservation. We consider an initial data with finite energy:

$$\begin{aligned} \mathcal{E}_0 = &\underbrace{\frac{c^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_y\Psi_0(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\Psi_1(x,y)|^2\,\mathrm{d}y\,\mathrm{d}x}_{\mathcal{E}_0^{\mathrm{vib}}} \\ &+\underbrace{\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\left(\frac{|v|^2}{2}+V(x)+\Phi(0,x)\right)\mathrm{d}v\,\mathrm{d}x}_{\mathcal{E}_0^{\mathrm{part}}} \in (-\infty,+\infty). \end{aligned}$$

For the solutions constructed in Theorem 2.3.3, we have seen that the self–consistent potential remains smooth enough so that the characteristic curves $t \mapsto (X(t), \xi(t))$ are well–defined. Therefore, we can write

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v) \left(\frac{|v|^2}{2} + V(x) + \Phi(t,x) \right) \mathrm{d}v \, \mathrm{d}x$$
$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v) \left(\frac{|\xi(t)|^2}{2} + V(X(t)) + \Phi(t, X(t)) \right) \mathrm{d}v \, \mathrm{d}x.$$

For any (t,x,v) we have the following equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[V(X(t)) + \Phi(t, X(t)) + \frac{|\xi(t)|^2}{2} \right] = (\partial_t \Phi)(t, X(t)).$$

Therefore, we get

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v) \left(\frac{|v|^2}{2} + V(x) + \Phi(t,x) \right) \mathrm{d}v \, \mathrm{d}x$$
$$= \mathcal{E}_0^{\text{part}} + \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v) \int_0^t (\partial_t \Phi)(s, X(s)) \, \mathrm{d}s \, \mathrm{d}v \, \mathrm{d}x$$
$$= \mathcal{E}_0^{\text{part}} + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} f(s,x,v) (\partial_t \Phi)(s,x) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}s$$
$$= \mathcal{E}_0^{\text{part}} + \int_0^t \int_{\mathbb{R}^d} \rho(s,x) (\partial_t \Phi)(s,x) \, \mathrm{d}x \, \mathrm{d}s.$$

Next, let Ψ be the unique solution of (2.4) associated to f. We first assume that the initial data Ψ_0 et Ψ_1 are smooth, say in $L^2(\mathbb{R}^d, H^2(\mathbb{R}^n))$. Therefore, going back to (2.8), we can check that Ψ lies in $C([0,\infty); L^2(\mathbb{R}^d, H^2(\mathbb{R}^n)))$. Integrations by parts lead to

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\partial_t \Psi(t,x,y)|^2 \, \mathrm{d}y \, \mathrm{d}x + \frac{c^2}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\nabla_y \Psi(t,x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y \right]$$
$$= \int_{\mathbb{R}^d \times \mathbb{R}^n} \partial_t \Psi \left(\partial_t^2 \Psi - c^2 \Delta_y \Psi \right) t,x,y) \, \mathrm{d}y \, \mathrm{d}x$$
$$= - \int_{\mathbb{R}^d \times \mathbb{R}^n} \partial_t \Psi(t,x,y) \; \rho(t,\cdot) \underset{x}{*} \sigma_1(x) \; \sigma_2(y) \, \mathrm{d}y \, \mathrm{d}x$$
$$= - \int_{\mathbb{R}^d} \rho \partial_t \Phi(t,x) \, \mathrm{d}x.$$

Hence, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\partial_t \Psi(t,x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y + \frac{c^2}{2} \int_{\mathbb{R}^d \times \mathbb{R}^n} |\nabla_y \Psi(t,x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y$$
$$= \mathcal{E}_0^{\text{vib}} - \int_0^t \int_{\mathbb{R}^d} \rho(s,x) (\partial_t \Phi)(s,x) \, \mathrm{d}x \, \mathrm{d}s.$$

It proves the energy conservation for such smooth data.

We go back to general data with finite energy: $\Psi_0 \in L^2(\mathbb{R}^d, H^1(\mathbb{R}^n))$ and $\Psi_1 \in L^2(\mathbb{R}^d \times \mathbb{R}^n)$. We approximate the data by Ψ_0^k and Ψ_1^k lying in $L^2(\mathbb{R}^d, H^2(\mathbb{R}^n))$. Using (2.8), one sees the associated sequence $(\Psi^k)_{k\in\mathbb{N}}$ of solutions to (2.4) converges to Ψ in $C([0,\infty); L^2(\mathbb{R}^d, H^1(\mathbb{R}^n)))$ and $C^1([0,\infty); L^2(\mathbb{R}^d \times \mathbb{R}^n))$. This implies one can pass to the limit in the energy conservation. ■

Remark 2.3.8 *We point out that, whereas energy conservation is an important physical property, it was not used here in the existence proof. In particular, one should notice that it does not provide directly useful a priori estimates on the kinetic energy, since the potential energy associated to the external potential V can be negative and unbounded under our assumptions. In order to deduce a useful estimate the assumptions on the initial data need to be strengthened: in addition to* (**H5**) *we suppose*

$$M_2 := \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)|x|^2 \,\mathrm{d}v\,\mathrm{d}x < \infty.$$

We set $V_-(x) = \max(-V(x), 0) \geq 0$. *Then* (**H2**) *implies*

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)V_-(x)\,\mathrm{d}v\,\mathrm{d}x &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)C(1+|x|^2)\,\mathrm{d}v\,\mathrm{d}x \\ &\leq C\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} + C\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)|X(t)|^2\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

where $X(t)$ stand for the first (space) component of $\varphi_0^t(x,v)$. Reproducing the estimates of the proof of Lemma 2.3.1, we get

$$|X(t)| \leq |x|e^{\sqrt{2C}t} + \frac{1}{\sqrt{C}}\left(V(x) + \frac{|v|^2}{2} + \Phi(0,x)\right)^{1/2}(e^{\sqrt{2C}t} - 1) + b(t)$$

where

$$b(t) = \sqrt{2}\int_0^t \Big(C + \|\Phi(s,\cdot)|_{L^\infty(\mathbb{R}^d)} + s\|\partial_t\Phi\|_{C([0,s];L^\infty(\mathbb{R}^d))}\Big)^{1/2} e^{\sqrt{2C}(t-s)}\,\mathrm{d}s.$$

It follows that

$$|X(t)| \leq 9|x|^2 e^{2\sqrt{2C}t} + \frac{9}{C}\left(V(x) + \frac{|v|^2}{2} + \Phi(0,x)\right)(e^{\sqrt{2C}t} - 1)^2 + 9b(t)^2.$$

Eventually, we find

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)V_-(x)\,\mathrm{d}v\,\mathrm{d}x \leq Ce^{2\sqrt{2C}t}M_2 + 9(e^{\sqrt{2C}t}-1)^2\mathcal{E}_0 + C(9b(t)^2+1)\|f_0\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}.$$

Therefore the potential energy associated to the external potential cannot be too negative and all terms in the energy balance remain bounded on any finite time interval.

2.4 Large wave speed asymptotics

This section is devoted to the asymptotics of large wave speeds. Namely, we consider the following rescaled version of the system:

$$\begin{cases} \partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon - \nabla_x(V+\Phi_\epsilon)\cdot\nabla_v f_\epsilon = 0, \\ \Phi_\epsilon(t,x,y) = \displaystyle\int_{\mathbb{R}^n\times\mathbb{R}^d} \Psi_\epsilon(t,z,y)\sigma_2(y)\sigma_1(x-z)\,\mathrm{d}z\,\mathrm{d}y, \\ \Big(\partial^2_{tt} - \dfrac{1}{\epsilon}\Delta_y\Big)\Psi_\epsilon(t,x,y) = -\dfrac{1}{\epsilon}\sigma_2(y)\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d}\sigma_1(x-z)f(t,z,v)\,\mathrm{d}v\,\mathrm{d}z, \end{cases} \tag{2.15}$$

completed with suitable initial conditions. We are interested in the behavior of the solutions as $\epsilon\to 0$. We shall discuss below the physical meaning of this regime. But, let us first explain on formal grounds what can be expected. As $\epsilon\to 0$ the wave equation degenerates to

$$-\Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \sigma_1 \underset{x}{*} \rho(t,x), \qquad \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v.$$

We obtain readily the solution by uncoupling the variables:

$$\Psi(t,x,y) = \gamma(y)\ \sigma_1 \underset{x}{*} \rho(t,x)$$

where γ satisfies the mere Poisson equation $\Delta_y\gamma = \sigma_2$. At leading order the potential then becomes

$$\Phi(t,x) = -\kappa\ \Sigma \underset{x}{*} \rho(t,x), \qquad \Sigma = \sigma_1 * \sigma_1, \qquad \kappa = -\int_{\mathbb{R}^n} \sigma_2\gamma\,\mathrm{d}y.$$

Therefore, we guess that the limiting behavior is described by the following Vlasov equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x(V+\Phi)\cdot\nabla_v f = 0.$$

As far as the integration by parts makes sense (we shall see that difficulties in the analysis precisely arise when $n\le 2$), we observe that

$$\kappa = \int_{\mathbb{R}^n} |\nabla_y\gamma|^2\,\mathrm{d}y > 0.$$

It is then tempting to make the form function σ_1 depend on ϵ too, so that Σ resembles the kernel of $(-\Delta_x)$. We would arrive at the Vlasov–Poisson system, in the case of attractive forces. We wish to justify these asymptotic behaviors.

Dimension analysis

In (2.2), f is the density of particles in phase space: it gives a number of particles per unit volume of phase space. Let $T, L, \mathcal{V}$ be units for time, space and velocity respectively, and set

$$t' = t/T, \quad x' = x/L, \quad v' = v/\mathcal{V}$$

which define dimensionless quantities. Then, we set

$$f'(t',x',v')\ L^{-d}\ \mathcal{V}^{-d} = f(t,x,v)$$

(or maybe more conveniently $f'(t',x',v')\,\mathrm{d}v'\,\mathrm{d}x' = f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$). The external and interaction potential, V and Φ, have both the dimension of a velocity squared. We set

$$V(x) = \mathcal{V}_{\mathrm{ext}}^2\ V'(x'), \qquad \Phi(t,x) = \mathcal{W}^2\ \Phi'(t',x'),$$

where $\mathcal{V}_{\mathrm{ext}}$ and $\mathcal{W}$ thus have the dimension of a velocity. We switch to the dimensionless equation

$$\partial_{t'} f' + \frac{\mathcal{V}T}{L} v' \cdot \nabla_{x'} f' - \frac{T}{L\mathcal{V}} \mathcal{V}^2 \nabla_{x'} \Big(V' + \Big(\frac{\mathcal{W}}{\mathcal{V}}\Big)^2 \Phi' \Big) \cdot \nabla_{v'} f' = 0.$$

The definition of the interaction potential Φ is driven by the product $\sigma_2(z)\sigma_1(x)\,\mathrm{d}x$. We scale it as follows

$$\sigma_2(z)\sigma_1(x)\,\mathrm{d}x = \Sigma_\star L^d \sigma_2'(z')\sigma_1'(x')\,\mathrm{d}x'.$$

It might help the intuition to think z as a length variable, and thus c has a velocity, but there is not reason to assume such privileged units. Thus, we keep a general approach. For the vibrating field, we set

$$\psi(t,x,z) = \Psi_\star\ \psi'(t',x',z'), \qquad z' = z/\ell,$$

still with the convention that primed quantities are dimensionless. Accordingly, we obtain

$$\mathcal{W}^2 = \Sigma_\star L^d \Psi_\star \ell^n$$

and the consistent expression of the dimensionless potential

$$\Phi'(t',x') = \int \sigma_1'(x'-y')\sigma_2'(z')\psi(t',y',z')\,\mathrm{d}z'\,\mathrm{d}y'.$$

The wave equation becomes

$$\partial^2_{t't'}\psi' - \frac{T^2c^2}{\ell^2}\Delta_{z'}\psi' = -\underbrace{\frac{T^2\Sigma_\star L^d}{\Psi_\star} L^{-d}}_{\dfrac{T^2\Sigma_\star}{\Psi_\star}}\ \sigma_2'(z') \int \sigma_1'(x'-y') f'(t',y',v')\,\mathrm{d}v'\,\mathrm{d}y'.$$

Note that

$$\frac{T^2\Sigma_\star}{\Psi_\star} = \Sigma_\star L^d \ell^n \Psi_\star\ \frac{T^2}{\Psi_\star^2 L^d \ell^n} = \mathcal{W}^2\ \frac{T^2}{\Psi_\star^2 L^d \ell^n}.$$

Let us consider the energy balance where the following quantities, all having the homogeneity of a velocity squared, appear:

- the kinetic energy of the particles $\int v^2 f\,\mathrm{d}v\,\mathrm{d}x$; it scales like $\mathcal{V}^2$,
- the external potential energy $\int V f\,\mathrm{d}v\,\mathrm{d}x$; it scales like $\mathcal{V}_{\mathrm{ext}}^2$,
- the coupling energy $\int \Phi f\,\mathrm{d}v\,\mathrm{d}x$; it scales like $\mathcal{W}^2$,

- the wave energy which splits into:

  a) $\int |\partial_t \psi|^2 \, \mathrm{d}z \, \mathrm{d}x$, which scales like $\Psi_\star^2 \frac{L^d \ell^n}{T^2}$,

  b) $c^2 \int |\nabla_z \partial_t \psi|^2 \, \mathrm{d}z \, \mathrm{d}x$, which scales like $c^2 \Psi_\star^2 \frac{L^d \ell^n}{\ell^2}$.

  Note that the kinetic energy in a) is $\frac{\ell^2}{c^2 T^2}$ times the elastic energy in b).

To recap, we have at hand 5 parameters imposed by the model $L, \ell, c, \mathcal{W}, \Sigma_\star$ and two parameters governed by the initial conditions $\mathcal{V}$ and $\Psi_\star$. They define the five energies described above.

We turn to the scaling assumptions. It is convenient to think of them by comparing the different time scales involved in the equations. We set

$$\epsilon = \left(\frac{\ell}{cT}\right)^2 \ll 1.$$

If ℓ is a length, say the size of the support of the source σ_2, then this regime means that the time a typical particle needs to cross L (the support of σ_1) is much longer than the time the wave needs to cross ℓ (the support of σ_2). Next we suppose that the kinetic energy of the particle, the energy of the particle associated to the external potential, the elastic energy of the wave as well as the interaction energy, all have the same strength, which can expressed by setting

$$\frac{L}{T} = \mathcal{V} = \mathcal{V}_{\text{ext}} = \mathcal{W} = \sqrt{c^2 \Psi_\star^2 L^d \ell^{n-2}}.$$

As a consequence, it imposes the following scaling of the coupling constant

$$\frac{\Psi_\star}{T^2 \Sigma_\star} = \epsilon.$$

It also means that the kinetic energy of the wave is small with respect to its elastic energy.

Statements of the results

Throughout this Section, we assume (**H1**), and we shall strengthen the assumptions (**H2**)–(**H5**) as follows (note that since we are dealing with sequences of initial data, it is important to make the estimates uniform with respect to the scaling parameter):

$$\text{the external potential } V \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^d) \text{ is non negative,} \tag{H7}$$

$$\left.\begin{array}{l} f_{0,\epsilon} \in L^1(\mathbb{R}^d \times \mathbb{R}^d), \text{ with a uniformly bounded norm,} \\ \text{and } \Psi_{0,\epsilon}, \Psi_{1,\epsilon} \in L^2(\mathbb{R}^d \times \mathbb{R}^n) \text{ are such that the rescaled initial energy} \\ \mathcal{E}_{0,\epsilon} = \displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{v^2}{2} + V + |\Phi_\epsilon|\right) f_{0,\epsilon} \, \mathrm{d}v \, \mathrm{d}x \\ \qquad\qquad + \displaystyle\frac{\epsilon}{2} \int_{\mathbb{R}^n \times \mathbb{R}^d} |\Psi_{1,\epsilon}|^2 \, \mathrm{d}y \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^d} |\nabla_y \Psi_{0,\epsilon}|^2 \, \mathrm{d}y \, \mathrm{d}x \\ \text{is uniformly bounded: } 0 \leq \sup_{\epsilon > 0} \mathcal{E}_{0,\epsilon} = \bar{\mathcal{E}}_0 < \infty. \end{array}\right\} \tag{H8}$$

$$f_{0,\epsilon} \text{ is bounded in } L^\infty(\mathbb{R}^d \times \mathbb{R}^d). \tag{\textbf{H9}}$$

Theorem 2.4.1 *Suppose $n \geq 3$. Let* (**H1**) *and* (**H7**)–(**H9**) *be satisfied. Let $(f_\epsilon, \Psi_\epsilon)$ be the associated solution to* (2.15)*. Then, there exists a subsequence such that f_ϵ converges in $C([0,T]; L^p((\mathbb{R}^d \times \mathbb{R}^d) - weak))$ for any $1 \leq p < \infty$ to f solution of the following Vlasov equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x(V + \bar{\Phi}) \cdot \nabla_v f = 0, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{2.16}$$

where

$$\bar{\Phi} = -\kappa \Sigma * \rho, \qquad \Sigma = \sigma_1 \underset{x}{*} \sigma_1, \qquad \kappa = \int_{\mathbb{R}^n} \frac{|\widehat{\sigma_2}(\xi)|^2}{(2\pi)^n |\xi|^2} \, \mathrm{d}\xi,$$

and f_0 is the weak limit in $L^p(\mathbb{R}^d \times \mathbb{R}^d)$ of $f_{0,\epsilon}$.

In order to derive the Vlasov–Poisson system from (2.15), the form function σ_1 need to be appropriately defined and scaled with respect to ϵ. Let θ and δ be two radially symmetric functions in $C_c^\infty(\mathbb{R}^d)$ verifying:

$$0 \leq \theta, \delta \leq 1 \qquad \theta(x) = 1 \text{ for } |x| \leq 1, \qquad \theta(x) = 0 \text{ for } |x| \geq 2, \qquad \int_{\mathbb{R}^d} \delta(x) \, \mathrm{d}x = 1.$$

We set $\theta_\epsilon(x) = \theta(\sqrt{\epsilon} x)$ et $\delta_\epsilon(x) = \dfrac{1}{\epsilon^{d/2}} \delta(x/\sqrt{\epsilon})$ and

$$\sigma_{1,\epsilon} = C_d \delta_\epsilon * \frac{\theta_\epsilon}{|\cdot|^{d-1}}, \qquad \text{with } C_d = \left(|\mathbb{S}^{d-1}| \int_{\mathbb{R}^d} \frac{\mathrm{d}x}{|x|^{d-1} |e_1 - x|^{d-1}} \right)^{-1/2}. \tag{2.17}$$

Theorem 2.4.2 *Let $d = 3$ and $n \geq 3$. Assume* (**H1**) *and* (**H7**)–(**H9**)*. Let $(f_\epsilon, \Psi_\epsilon)$ be the associated solution to* (2.15) *with $\sigma_1 = \sigma_{1,\epsilon}$ depending on ϵ according to* (2.17)*. Then, there exists a subsequence such that f_ϵ converges in $C([0,T]; L^p(\mathbb{R}^3 \times \mathbb{R}^3) - weak)$ for any $1 < p < \infty$ to f solution of the attractive Vlasov–Poisson equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x(V + \bar{\Phi}) \cdot \nabla_v f = 0, \\ \Delta \bar{\Phi} = \kappa \rho, \\ f(0,x,v) = f_0(x,v) \end{cases} \tag{2.18}$$

where f_0 is the weak limit in $L^p(\mathbb{R}^3 \times \mathbb{R}^3)$ of $f_{0,\epsilon}$.

Remark 2.4.3 *In Theorem 2.4.1, if, furthermore, we assume that $\Big(f_{0,\epsilon}\Big)_{\epsilon>0}$ converge (in the appropriate weak sense) to f_0, by uniqueness of the solution of the limit equation, the entire sequence $\Big(f_\epsilon\Big)_{\epsilon>0}$ converges to f. For Theorem 2.4.1 and Theorem 2.4.2, if the initial data converges strongly to f_0 in $L^p(\mathbb{R}^d \times \mathbb{R}^d)$, $1 \leq p < \infty$, then f_ϵ converges to f in $C([0,T]; L^p(\mathbb{R}^d \times \mathbb{R}^d))$.*

Convergence to the Vlasov equation with a smooth convolution kernel

Taking into account the rescaling, the analog of (2.7) for (2.15) reads

$$\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon - \nabla_x\Big(V + \Phi_{0,\epsilon} - \frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)\Big)\cdot\nabla_v f_\epsilon = 0, \tag{2.19}$$

with

$$\Phi_{0,\epsilon}(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \widetilde{\Psi}_\epsilon(t,z,y)\sigma_1(x-z)\sigma_2(y)\,\mathrm{d}y\,\mathrm{d}z.$$

where $\widetilde{\Psi}_\epsilon$ stands for the unique solution of the free linear wave equation (in $\mathbb{R}^n$) with wave speed $1/\epsilon$ and initial data $\Psi_{0,\epsilon}$ and $\Psi_{1,\epsilon}$, and

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)(t,x) &= \frac{1}{\epsilon}\int_{\mathbb{R}^d}\Sigma(x-z)\Big(\int_0^t \rho_\epsilon(t-s,z)\\ &\qquad\times\Big(\int_{\mathbb{R}^n}\frac{\sin(|\xi|s/\sqrt{\epsilon})}{|\xi|/\sqrt{\epsilon}}\;|\widehat{\sigma_2}(\xi)|^2\frac{\mathrm{d}\xi}{(2\pi)^n}\Big)\,\mathrm{d}s\Big)\,\mathrm{d}z \\ &= \Big(\Sigma\underset{x}{*}\int_0^{t/\sqrt{\epsilon}}\rho_\epsilon(t-s\sqrt{\epsilon},\cdot)\;q(s)\,\mathrm{d}s\Big)(x)\end{aligned}\tag{2.20}$$

where we have set

$$q(t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\sin(t|\xi|)}{|\xi|}|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi.$$

(it is nothing but $p(t)$ as introduced in Section 2.2 evaluated with $c=1$; of course when $c=1$ and $\epsilon=1$, the operators $\frac{1}{\epsilon}\mathcal{L}_\epsilon$ in (2.20) and $\mathcal{L}$ in (2.6) coincide.)

Lemma 2.4.4 *Let $n\geq 3$. Then q is integrable over $[0,+\infty[$ with*

$$\int_0^\infty q(t)\,\mathrm{d}t = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{|\widehat{\sigma_2}(\xi)|^2}{|\xi|^2}\,\mathrm{d}\xi := \kappa > 0.$$

Proof. By virtue of the dominated convergence theorem, $t\mapsto q(t)$ is continuous on $[0,\infty)$. Bearing in mind that σ_2 is radially symmetric, integrations by parts yield

$$\begin{aligned} q(t) &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty \sin(tr)r^{n-2}|\widehat{\sigma_2}(re_1)|^2\,\mathrm{d}r\\ &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty\frac{\cos(tr)}{t}\frac{\mathrm{d}}{\mathrm{d}r}\Big[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\Big]\,\mathrm{d}r\\ &= -\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty\frac{\sin(tr)}{t^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\Big]\,\mathrm{d}r.\end{aligned}$$

Hence, we can estimate as follows

$$|q(t)|\leq\frac{K}{t^2}\quad\text{ with }\quad K = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty\left|\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Big[r^{n-2}|\widehat{\sigma_2}(re_1)|^2\Big]\right|\,\mathrm{d}r<\infty$$

which proves $q \in L^1([0,\infty))$.
Next, we compute the integral of q. For $M > 0$ we get:

$$\begin{aligned}
\int_0^M q(t)\,\mathrm{d}t &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(\int_0^M \frac{\sin(t|\xi|)}{|\xi|}\,\mathrm{d}t\right)|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{1-\cos(M|\xi|)}{|\xi|^2}|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi \\
&= \kappa - \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n}\int_0^\infty \cos(Mr)r^{n-3}|\widehat{\sigma_2}(re_1)|^2\,\mathrm{d}r \\
&= \kappa - \frac{|\mathbb{S}^{n-1}|}{M(2\pi)^n}\int_0^\infty \sin(Mr)\frac{\mathrm{d}}{\mathrm{d}r}\left[r^{n-3}|\widehat{\sigma_2}(re_1)|^2\right]\,\mathrm{d}r.
\end{aligned}$$

We conclude by letting M tend to ∞.
Note that κ is infinite for $n=2$ since $\frac{|\sigma_2(\xi)|^2}{|\xi|^2} \sim_{\xi\to 0} \|\sigma_2\|^2_{L^1(\mathbb{R}^2)}\frac{1}{|\xi|^2}$ does not belong to $L^1(B(0,a))$ for any $a > 0$. ■

We turn to the proof of Theorem 2.4.1. Of course we have

$$\sup_{\epsilon>0}\|f_\epsilon(t,\cdot)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} = \sup_{\epsilon>0}\|f_{0,\epsilon}\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} := M_0 < \infty,$$

and the L^p norms

$$\|f_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} = \|f_{0,\epsilon}\|_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}$$

are also bounded, for any $1 \le p \le \infty$ by virtue of (**H9**). Furthermore, the energy conservation yields

$$\begin{aligned}
\mathcal{E}_\epsilon(t) = \int_{\mathbb{R}^d\times\mathbb{R}^d}&\left(\frac{v^2}{2}+V+\Phi_\epsilon\right)f_\epsilon\,\mathrm{d}v\,\mathrm{d}x \\
&+\frac{\epsilon}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\partial_t\Psi_\epsilon|^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\nabla_y\Psi_\epsilon|^2\,\mathrm{d}y\,\mathrm{d}x \le \bar{\mathcal{E}}_0.
\end{aligned}$$

Let us set

$$\mathcal{E}^{\mathrm{vib}}_{0,\epsilon} = \frac{\epsilon}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\Psi_{1,\epsilon}|^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\nabla_y\Psi_{0,\epsilon}|^2\,\mathrm{d}y\,\mathrm{d}x.$$

As a consequence of (**H1**) and (**H8**), $\mathcal{E}^{\mathrm{vib}}_{0,\epsilon}$ is bounded uniformly with respect to ϵ. Owing to the standard energy conservation for the free linear wave equation, we observe that $\|\nabla_y\widetilde{\Psi}_\epsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^d\times\mathbb{R}^n))} \le (2\mathcal{E}^{\mathrm{vib}}_{0,\epsilon})^{1/2}$. Then Sobolev's embedding (mind the condition $n \ge 3$) allows us to deduce the following key estimate on $\widetilde{\Psi}_\epsilon$:

$$\|\widetilde{\Psi}_\epsilon\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^d;L^{2n/(n-2)}(\mathbb{R}^n)))} \le C\Big(\mathcal{E}^{\mathrm{vib}}_{0,\epsilon}\Big)^{1/2} \le C\Big(\bar{\mathcal{E}}_0\Big)^{1/2} \tag{2.21}$$

Applying Hölder inequalities, we are thus led to:

$$|\Phi_{0,\epsilon}(t,x)| \le C\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\bar{\mathcal{E}}_0\Big)^{1/2}, \tag{2.22}$$

and similarly

$$|\nabla_x \Phi_{0,\epsilon}(t,x)| \leq C\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\bar{\mathcal{E}}_0\Big)^{1/2}. \tag{2.23}$$

Concerning the asymptotic behavior, we shall use the following claim. It is not a direct consequence of these estimates and it will be justified later on.

Lemma 2.4.5 *Let $\chi \in C_c^\infty([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. Then, we have*

$$\lim_{\epsilon\to 0}\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \nabla_x\Phi_{0,\epsilon}\chi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The cornerstone of the proof of Theorem 2.4.1 is the estimate of the self–consistent potential. By virtue of (2.20), for any $1\leq p\leq\infty$ we get

$$\begin{aligned}\Big\|\frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)(t,\cdot)\Big\|_{L^p(\mathbb{R}^d)} &\leq \|\Sigma\|_{L^p(\mathbb{R}^d)}\|\rho_\epsilon\|_{L^\infty([0,\infty),L^1(\mathbb{R}^d))}\int_0^\infty|q(s)|\,\mathrm{d}s\\ &\leq \|\Sigma\|_{L^p(\mathbb{R}^d)}M_0\|q\|_{L^1([0,+\infty))},\end{aligned}$$

as well as

$$\Big\|\frac{1}{\epsilon}\nabla_x\mathcal{L}_\epsilon(f_\epsilon)(t,\cdot)\Big\|_{L^p(\mathbb{R}^d)} \leq \|\nabla_x\Sigma\|_{L^p(\mathbb{R}^d)}M_0\|q\|_{L^1([0,+\infty))}.$$

Let $\chi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. We have

$$\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big| \leq M_0\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}$$

and

$$\begin{aligned}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x\Big| &\leq M_0\|v\cdot\nabla\chi-\nabla V\cdot\nabla_v\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\ +\Big(\|q\|_{L^1([0,+\infty))}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}M_0^2 &+ CM_0\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\bar{\mathcal{E}}_0\Big)^{1/2}\Big)\\ &\times\|\nabla_v\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.\end{aligned}$$

Reproducing arguments detailed in the previous Section, we deduce that we can assume, possibly at the price of extracting a subsequence, that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x$$

holds for any $\chi\in L^{p'}(\mathbb{R}^d\times\mathbb{R}^d)$ uniformly on $[0,T]$, $0<T<\infty$, with $f\in C([0,T];L^p(\mathbb{R}^d\times\mathbb{R}^d)-\text{weak})$, $1<p<\infty$, $1/p+1/p'=1$.

Next, we establish the tightness of $\Big(f_\epsilon\Big)_{\epsilon>0}$ with respect to the velocity variable, which will be necessary to show that the macroscopic density ρ_ϵ passes to the limit. Since $\Phi_{0,\epsilon}$

and $\frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)$ are uniformly bounded and $V \geq 0$, we infer from the energy conservation the estimate

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|v|^2}{2} f_\epsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\ \leq \bar{\mathcal{E}}_0 + \|q\|_{L^1([0,+\infty))}\|\Sigma\|_{L^\infty(\mathbb{R}^d)}M_0^2 + CM_0\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\bar{\mathcal{E}}_0\Big)^{1/2}.$$

Hence, we can check that $\rho_\epsilon(t,x) = \int_{\mathbb{R}^d} f_\epsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$ satisfies

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\rho_\epsilon(t,x)\chi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\rho(t,x)\chi(x)\,\mathrm{d}x \tag{2.24}$$

for any $\chi \in C_0(\mathbb{R}^d)$, with $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. As a matter of fact, we note that (**H1**) and (2.24) imply

$$\lim_{\epsilon\to 0}\nabla_x\Sigma * \rho_\epsilon(t,x) = \nabla_x\Sigma * \rho(t,x) \quad \text{for any } (t,x)\in[0,T]\times\mathbb{R}^d. \tag{2.25}$$

Furthermore, we have

$$|D_x^2(\Sigma * \rho_\epsilon)(t,x)| \leq M_0\ \|\Sigma\|_{W^{2,\infty}(\mathbb{R}^d)},$$

and, by using mass conservation and the Cauchy-Schwarz inquality,

$$\begin{aligned}
|\partial_t(\nabla_x\Sigma * \rho_\epsilon)(t,x)| &= \left|\int_{\mathbb{R}^d} D_x^2\Sigma(x-y)\left(\int_{\mathbb{R}^d} v f_\epsilon(t,y,v)\,\mathrm{d}v\right)\,\mathrm{d}y\right| \\
&\leq \|\Sigma\|_{W^{2,\infty}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} v^2 f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2} \\
&\leq \|\Sigma\|_{W^{2,\infty}(\mathbb{R}^d)}\ \sqrt{2M_0}\ \Big(\bar{\mathcal{E}}_0 + \|q\|_{L^1([0,+\infty))}\|\Sigma\|_{L^\infty(\mathbb{R}^d)}M_0^2 \\
&\qquad\qquad + CM_0\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\bar{\mathcal{E}}_0\Big)^{1/2}\Big)^{1/2}.
\end{aligned}$$

Therefore convergence (2.25) holds uniformly on any compact set of $[0,\infty)\times\mathbb{R}^d$.

We turn to examine the convergence of $\frac{1}{\epsilon}\nabla_x \mathcal{L}_\epsilon(f_\epsilon)$ to $\kappa \nabla_x \Sigma * \rho$. We have

$$\begin{aligned}
&\left|\frac{1}{\epsilon}\nabla_x \mathcal{L}_\epsilon(f_\epsilon)(t,x) - \kappa\nabla_x\Sigma * \rho(t,x)\right| \\
&\quad = \left|\int_0^{t/\sqrt{\epsilon}} \nabla_x \Sigma * \rho_\epsilon(t - s\sqrt{\epsilon}, x) q(s)\,\mathrm{d}s - \kappa \nabla_x \Sigma * \rho(t,x)\right| \\
&\quad \le \left|\int_0^{t/\sqrt{\epsilon}} \Big(\nabla_x \Sigma * \rho_\epsilon(t - s\sqrt{\epsilon}, x) - \nabla_x \Sigma * \rho(t,x)\Big) q(s)\,\mathrm{d}s\right| \\
&\qquad + \left|\int_{t/\sqrt{\epsilon}}^\infty q(s)\,\mathrm{d}s\right| \|\nabla_x \Sigma * \rho\|_{L^\infty((0,\infty)\times\mathbb{R}^d)} \\
&\quad \le \int_0^{t/\sqrt{\epsilon}} |(\nabla_x\Sigma * \rho_\epsilon - \nabla_x \Sigma * \rho)(t - s\sqrt{\epsilon}, x)|\ |q(s)|\,\mathrm{d}s \\
&\qquad + \int_0^{t/\sqrt{\epsilon}} |\nabla_x \Sigma * \rho(t - s\sqrt{\epsilon}, x) - \nabla_x \Sigma * \rho(t,x)|\ |q(s)|\,\mathrm{d}s \\
&\qquad + \int_{t/\sqrt{\epsilon}}^\infty |q(s)|\,\mathrm{d}s\ \|\nabla_x\Sigma * \rho\|_{L^\infty((0,\infty)\times\mathbb{R}^d)}.
\end{aligned}$$

Let us denote by $\mathrm{I}_\epsilon(t,x)$, $\mathrm{II}_\epsilon(t,x)$, $\mathrm{III}_\epsilon(t)$, the three terms of the right hand side. Firstly, for any $t > 0$, $\mathrm{III}_\epsilon(t)$ tends to 0 as $\epsilon \to 0$, and it is dominated by $\kappa\|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)} M_0$. Secondly, for any $0 < T < \infty$ and any compact set $K \subset \mathbb{R}^d$, when (t,x) lies in $[0,T] \times K$, we can estimate

$$|\mathrm{I}_\epsilon(t,x)| \le \|\nabla_x \Sigma * \rho_\epsilon - \nabla_x \Sigma * \rho\|_{L^\infty([0,T]\times K)} \|q\|_{L^1([0,\infty)}$$

which also goes to 0 as $\epsilon \to 0$. Eventually, still considering $(t,x) \in [0,T] \times K$, we write

$$|\mathrm{II}_\epsilon(t,x)| \le \int_0^{t/\sqrt{\epsilon}} \sup_{z\in K} |\nabla_x \Sigma * \rho(t - s\sqrt{\epsilon}, z) - \nabla_x\Sigma * \rho(t,z)|\ |q(s)|\,\mathrm{d}s.$$

By using the Lebesgue theorem, we justify that it tends to 0 as $\epsilon \to 0$ since $(t,x) \mapsto \nabla_x \Sigma * \rho(t,x)$ is uniformly continuous over any compact set, the integrand is dominated by $2\|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)} M_0 |q(s)|$, and $q \in L^1([0,\infty))$. Therefore, for any $0 < t < T < \infty$ and any compact set $K \subset \mathbb{R}^d$,

$$\sup_{x\in K}\left|\frac{1}{\epsilon}\nabla_x \mathcal{L}_\epsilon(f_\epsilon) - \kappa \nabla_x \Sigma * \rho\right|(t,x) \xrightarrow[\epsilon\to 0]{} 0,$$

and this quantity is bounded uniformly with respect to $0 \le t \le T < \infty$ and $\epsilon > 0$.

We go back to the weak formulation of (2.15). Let $\chi \in C_c^\infty([0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. We suppose that $\mathrm{supp}(\chi) \subset [0,T] \times \bar{B}(0,M) \times \bar{B}(0,M)$. We have

$$\begin{aligned}
&-\int_{\mathbb{R}^d\times\mathbb{R}^d} f_{0,\epsilon}\chi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \partial_t \chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon v \cdot \nabla_x \chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} \int f_\epsilon \nabla_v \chi \cdot \nabla_x (V + \Phi_{0,\epsilon})\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad = \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon\ \nabla_x \frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon) \cdot \nabla_v \chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Obviously, there is no difficulty with the linear terms of the left hand side. For the non linear term we proceed as follows:

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \ \nabla_x \frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)\cdot\nabla_v\chi \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f \ \kappa\nabla_x\Sigma*\rho\cdot\nabla_v\chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
= \int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \left(\nabla_x\frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon) - \kappa\nabla_x\Sigma*\rho\right)\cdot\nabla_v\chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\
+\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} (f_\epsilon - f)\ \kappa\nabla_x\Sigma*\rho\cdot\nabla_v\chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The last term directly passes to the limit. The first integral in the right hand side is dominated by

$$M_0\|\nabla_v\chi\|_{L^\infty([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)} \ \int_0^T \sup_{y\in\bar{B}(0,M)}\left|\nabla_x\frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon) - \kappa\nabla_x\Sigma*\rho\right|(t,y)\,\mathrm{d}t.$$

We conclude by a mere application of the Lebesgue Theorem.

If the initial data $f_{0,\epsilon}$ converge strongly to f_0 in $L^p(\mathbb{R}^d\times\mathbb{R}^d)$, the nature of the convergence of f_ϵ to f can be improved by applying general stability results for transport equations, see [31, Th. II.4 & Th. II.5], or [16, Th. VI.1.9].

Proof of Lemma 2.4.5 As a matter of fact, the variable $x\in\mathbb{R}^d$ just appears as a parameter for the wave equation, and $\Upsilon_\epsilon(t,x,y) = (\sigma_1 * \widetilde{\Psi}_\epsilon(t,\cdot,y))(x)$ solves the linear wave equation

$$\epsilon\partial_{tt}^2\Upsilon_\epsilon - \Delta_y\Upsilon_\epsilon = 0,$$

with the data

$$\Upsilon_\epsilon(0,x,y) = \sigma_1 * \Psi_{0,\epsilon}(x,y), \qquad \partial_t\Upsilon_\epsilon(0,x,y) = \sigma_1*\Psi_{1,\epsilon}(x,y).$$

The parameter x being fixed, we appeal to the Strichartz estimate, see [69, Corollary 1.3] or [76, Theorem 4.2, for the case $n=3$],

$$\frac{1}{\epsilon^{1/(2p)}}\left(\int_0^\infty\left(\int_{\mathbb{R}^n}|\Upsilon_\epsilon(t,x,y)|^q\,\mathrm{d}y\right)^{p/q}\,\mathrm{d}t\right)^{1/p} \le C\sqrt{\mathscr{E}_{1,\epsilon}^{\mathrm{vib}}(x)}$$

where we set

$$\mathscr{E}_{1,\epsilon}^{\mathrm{vib}}(x) = \epsilon\int_{\mathbb{R}^n}|\sigma_1*\Psi_{1,\epsilon}(x,y)|^2\,\mathrm{d}y + \int_{\mathbb{R}^n}|\sigma_1*\nabla_y\Psi_{0,\epsilon}(x,y)|^2\,\mathrm{d}y.$$

(That $\frac{1}{\epsilon^{1/(2p)}}$ appears in the inequality can be checked by changing variables and observing that $\Upsilon_\epsilon(t\sqrt{\epsilon},x,y)$ satisfies the wave equation with speed equals to 1 and data $(\sigma_1*\Psi_{0\epsilon},\sqrt{\epsilon}\sigma_1*\Psi_{1,\epsilon})$.) This inequality holds for admissible exponents:

$$2\le p\le q\le\infty,\quad \frac{1}{p}+\frac{n}{q}=\frac{n}{2}-1,\quad \frac{2}{p}+\frac{n-1}{q}\le\frac{n-1}{2},\quad (p,q,n)\neq(2,\infty,3).$$

Observe that

$$\int_{\mathbb{R}^d} \mathscr{E}^{\mathrm{vib}}_{1,\epsilon}(x)\,\mathrm{d}x \le \|\sigma_1\|_{L^1(\mathbb{R}^d)}\ \mathscr{E}^{\mathrm{vib}}_{0,\epsilon} \le \|\sigma_1\|_{L^1(\mathbb{R}^d)}\ \bar{\mathscr{E}}_0.$$

It follows that

$$\int_{\mathbb{R}^d}\left(\int_0^\infty\left(\int_{\mathbb{R}^n}|\Upsilon_\epsilon(t,x,y)|^q\,\mathrm{d}y\right)^{p/q}\mathrm{d}t\right)^{2/p}\mathrm{d}x \le C^2\|\sigma_1\|_{L^1(\mathbb{R}^d)}\ \bar{\mathscr{E}}_0\epsilon^{1/p} \xrightarrow[\epsilon\to 0]{} 0.$$

A similar reasoning applies to $\nabla_x\Upsilon_\epsilon$ with $\nabla_x\sigma_1$ replacing σ_1. Let $\chi\in C^\infty_c([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. We suppose that $\mathrm{supp}(\chi)\subset\{0\le t\le M,\ |x|\le M,\ |v|\le M\}$ for some $0<M<\infty$. We are left with the task of estimating

$$\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon\nabla_x\Phi_{0,\epsilon}\chi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty\int_{\mathbb{R}^d} R_\epsilon(t,x)\nabla_x\Phi_{0,\epsilon}(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

where we have set

$$R_\epsilon(t,x) = \int_{\mathbb{R}^d} f_\epsilon\chi(t,x,v)\,\mathrm{d}v.$$

With the standard notation $1/p+1/p'=1$, using Hölder's inequality twice, we get

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon\nabla_x\Phi_{0,\epsilon}\chi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\quad\le\left(\int_{\mathbb{R}^d}\left(\int_0^\infty|R_\epsilon(t,x)|^{p'}\,\mathrm{d}t\right)^{2/p'}\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left(\int_0^\infty|\nabla_x\Phi_{0,\epsilon}(t,x)|^p\,\mathrm{d}t\right)^{2/p}\mathrm{d}x\right)^{1/2}.\end{aligned}$$

We readily obtain

$$\begin{aligned}\left(\int_{\mathbb{R}^d}\left(\int_0^\infty|R_\epsilon(t,x)|^{p'}\,\mathrm{d}t\right)^{2/p'}\mathrm{d}x\right)^{1/2} &\le M^{d+d/2+1/p'}\|f_\epsilon\chi\|_{L^\infty((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)}\\ &\le M^{d+d/2+1/p'}\|f_{0,\epsilon}\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)}\end{aligned}$$

which is thus bounded uniformly with respect to $\epsilon>0$. Furthermore, with $1/q+1/q'=1$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_0^\infty|\nabla_x\Phi_{0,\epsilon}(t,x)|^p\,\mathrm{d}t\right)^{2/p}\mathrm{d}x &= \int_{\mathbb{R}^d}\left(\int_0^\infty\left|\int_{\mathbb{R}^n}\sigma_2(y)\nabla_x\Upsilon_\epsilon(t,x,y)\,\mathrm{d}y\right|^p\mathrm{d}t\right)^{2/p}\mathrm{d}x\\ &\le\|\sigma_2\|_{L^{q'}(\mathbb{R}^d)}\int_{\mathbb{R}^d}\left(\int_0^\infty\left|\int_{\mathbb{R}^n}|\nabla_x\Upsilon_\epsilon(t,x,y)|^q\,\mathrm{d}y\right|^{p/q}\mathrm{d}t\right)^{2/p}\mathrm{d}x\end{aligned}$$

which tends to 0 like $\epsilon^{1/p}$. ∎

Convergence to the Vlasov–Poisson system

The existence theory for the Vlasov–Poisson system dates back to [7]; an overview of the features of both the repulsive or attractive cases can be found in the lecture notes [14]. The following statements are classical tools of this analysis, that will be useful for our purposes as well.

Lemma 2.4.6 (Interpolation estimates) *Let $f \in L^1 \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be such that $|v|^m f \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Then $\rho = \int_{\mathbb{R}^d} f \,\mathrm{d}v$ lies in $L^{(m+d)/d}(\mathbb{R}^d)$ with*

$$\|\rho\|_{L^{(d+m)/d}(\mathbb{R}^d)} \le C(m,d)\|f\|_{L^\infty}^{m/(d+m)(\mathbb{R}^d)} \left(\int |v|^m f \,\mathrm{d}v\,\mathrm{d}x\right)^{d/(d+m)}.$$

where $C(m,d) = 2|B(0,1)|^{m/(m+d)}$.

Lemma 2.4.7 (Hardy-Littlewood-Sobolev inequality) *Let $1 < p, r < \infty$ and $0 < \lambda < d$. Assume $1/p + 1/r = 2 - \lambda/d$. There exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^r(\mathbb{R}^d)$ we have*

$$\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{f(x)g(y)}{|x-y|^\lambda} \,\mathrm{d}y\,\mathrm{d}x \right| \le C\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^r(\mathbb{R}^d)}.$$

We refer the reader to [14, Lemma 3.4] and [62, Th. 4.3], respectively, for further details. Next, we check the convergence of the approximate kernel defined by $\sigma_{1,\epsilon}$.

Lemma 2.4.8 *Let $d \ge 3$. For any $d/(d-1) < q < \infty$, we have:*

$$\left\| \nabla \left(\frac{C_d \theta_\epsilon}{|\cdot|^{d-1}} * \frac{C_d \theta_\epsilon}{|\cdot|^{d-1}} \right)(x) + (d-2)\frac{x}{|\mathbb{S}^{d-1}||x|^d} \right\|_{L^q(\mathbb{R}^d)} \xrightarrow[\epsilon \to 0]{} 0.$$

Proof. We remind the reader that the convolution by $|x|^{1-d}$ is associated to the Fourier transform of the operator with symbol $1/|\xi|$, see [62, Th. 5.9]. The convolution of radially symmetric functions is radially symmetric too. For $d \ge 3$, we compute as follows

$$\begin{aligned}
\left(\frac{1}{|\cdot|^{d-1}} * \frac{1}{|\cdot|^{d-1}} \right)(x) &= \int_{\mathbb{R}^d} \frac{\mathrm{d}y}{|y|^{d-1}|x-y|^{d-1}} \\
&= \int_{\mathbb{R}^d} \frac{|x|^d \,\mathrm{d}y}{|x|^{d-1}|e_1 - y|^{d-1}|x|^{d-1}|y|^{d-1}} = \frac{1}{|\mathbb{S}^{d-1}|\ C_d^2\ |x|^{d-2}}.
\end{aligned}$$

Differentiating yields

$$\nabla \left(\frac{C_d}{|\cdot|^{d-1}} * \frac{C_d}{|\cdot|^{d-1}} \right)(x) = -\frac{d-2}{|\mathbb{S}|^{d-1}}\ \frac{x}{|x|^d}.$$

Hence, we can write

$$\begin{aligned}
\mathscr{O}_\epsilon(x) &:= \nabla \left(\frac{C_d \theta_\epsilon}{|\cdot|^{d-1}} * \frac{C_d \theta_\epsilon}{|\cdot|^{d-1}} \right)(x) + \frac{(d-2)x}{|\mathbb{S}^{d-1}||x|^d} \\
&= C_d^2 \nabla \left(\frac{\theta_\epsilon + 1}{|\cdot|^{d-1}} * \frac{\theta_\epsilon - 1}{|\cdot|^{d-1}} \right)(x) \\
&= C_d^2 \frac{\theta_\epsilon + 1}{|\cdot|^{d-1}} * \left(\frac{\nabla \theta_\epsilon}{|\cdot|^{d-1}} + (1-d)(\theta_\epsilon - 1)\frac{\cdot}{|\cdot|^{d+1}} \right)(x).
\end{aligned}$$

Let $p > 1$. On the one hand, we have

$$\begin{aligned}\left\|\frac{\nabla\theta_\epsilon}{|\cdot|^{d-1}}\right\|^p_{L^p(\mathbb{R}^d)} &= \int_{\mathbb{R}^d}\frac{|\nabla\theta_\epsilon(x)|^p}{|x|^{p(d-1)}}\,\mathrm{d}x\\ &\leq (\sqrt{\epsilon})^p\,\|\nabla\theta\|^p_{L^\infty(\mathbb{R}^d)}\int_{1\leq\sqrt{\epsilon}|x|\leq 2}\frac{\mathrm{d}x}{|x|^{p(d-1)}}\\ &\leq (\sqrt{\epsilon})^{d(p-1)}\|\nabla\theta\|^p_{L^\infty(\mathbb{R}^d)}\int_{1\leq|x|\leq 2}\frac{\mathrm{d}x}{|x|^{p(d-1)}}.\end{aligned}$$

On the other hand, we get

$$\int_{\mathbb{R}^d}\left|\frac{(\theta_\epsilon(x)-1)x}{|x|^{d+1}}\right|^p\mathrm{d}x \leq \int_{\sqrt{\epsilon}|x|\geq 1}\frac{\mathrm{d}x}{|x|^{pd}} = (\sqrt{\epsilon})^{d(p-1)}\left(\int_{|x|\geq 1}\frac{\mathrm{d}x}{|x|^{pd}}\right).$$

Accordingly, the following estimate holds:

$$\left\|\frac{\nabla\theta_\epsilon}{|\cdot|^{d-1}} + (1-d)\frac{(\theta_\epsilon-1)\cdot}{|\cdot|^{d+1}}\right\|_{L^p} \leq C\epsilon^{d(p-1)/(2p)}, \tag{2.26}$$

where $C > 0$ depends on p and d only. Finally we remark that $0 \leq \frac{\theta_\epsilon(x)+1}{|x|^{d-1}} \leq \frac{2}{|x|^{d-1}}$. By coming back to Lemma 2.4.7, we deduce that there exists a constant $\tilde{C} > 0$ such that

$$\left|\int_{\mathbb{R}^d}\mathscr{O}_\epsilon(x)g(x)\,\mathrm{d}x\right| \leq \tilde{C}\|g\|_{L^r(\mathbb{R}^d)}\,(\sqrt{\epsilon})^{d(p-1)/p}$$

holds for any $g \in L^r(\mathbb{R}^d)$, with $1/r = (d+1)/d - 1/p > 1/d$, $r > 1$. Therefore, by duality, it means that $\mathscr{O}_\epsilon$ converges to 0 in $L^q(\mathbb{R}^d)$ for any $d/(d-1) < q < \infty$. ■

Proof of Theorem 2.4.2. From now on, we restrict to the case of space dimension $d = 3$. Compared to the previous Section, additional difficulties come from the dependence of the form function σ_1 with respect to ϵ so that deducing uniform estimates from the energy conservation is not direct.

Step 1. Establishing uniform estimates.

We start by observing that f_ϵ is bounded in $L^\infty(0,\infty;L^p(\mathbb{R}^3\times\mathbb{R}^3))$ for any $1 \leq p \leq \infty$, since

$$\|f_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f_{0,\epsilon}\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}.$$

Next, the energy conservation becomes

$$\begin{aligned}\mathcal{E}_\epsilon(t) &= \frac{\epsilon}{2}\int_{\mathbb{R}^3\times\mathbb{R}^n}|\partial_t\Psi_\epsilon(t,x,y)|^2\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^n}|\nabla_y\Psi_\epsilon(t,x,y)|^2\,\mathrm{d}y\,\mathrm{d}x\\ &\quad+\int_{\mathbb{R}^3\times\mathbb{R}^3}f_\epsilon(t,x,v)\left(\frac{|v|^2}{2}+V(x)+\Phi_\epsilon(t,x)\right)\mathrm{d}v\,\mathrm{d}x\\ &= \mathcal{E}_\epsilon(0) \leq \bar{\mathcal{E}}_0.\end{aligned}$$

Let us study the coupling term:

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}f_\epsilon(t,x,v)\Phi_\epsilon(t,x)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^3}\rho_\epsilon(t,x)\Phi_\epsilon(t,x)\,\mathrm{d}x = \mathrm{S}_\epsilon(t)+\mathrm{T}_\epsilon(t)$$

where we have set

$$\begin{aligned}
\mathrm{S}_\epsilon(t) &= -\frac{1}{\epsilon}\int_{\mathbb{R}^3} \rho_\epsilon \mathcal{L}_\epsilon(f_\epsilon)(t,x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \Big(\sigma_{1,\epsilon} * \sigma_{1,\epsilon} * \int_0^{t/\sqrt{\epsilon}} q(s)\rho_\epsilon(t-s\sqrt{\epsilon},\cdot)\,\mathrm{d}s\Big)(x)\rho_\epsilon(t,x)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^3} \Big(\sigma_{1,\epsilon} * \int_0^{t/\sqrt{\epsilon}} q(s)\rho_\epsilon(t-s\sqrt{\epsilon},\cdot)\,\mathrm{d}s\Big)(x)\ \sigma_{1,\epsilon} * \rho_\epsilon(t,x)\,\mathrm{d}x
\end{aligned}$$

and

$$\mathrm{T}_\epsilon(t) = \int_{\mathbb{R}^3} \rho_\epsilon \Phi_{0,\epsilon}(t,x)\,\mathrm{d}x, \qquad \Phi_{0,\epsilon}(t,x) = \Big(\sigma_{1,\epsilon} * \int_{\mathbb{R}^n} \widetilde{\Psi}_\epsilon(t,\cdot,y)\sigma_2(y)\,\mathrm{d}y\Big)(x).$$

Like in the previous Section, $\widetilde{\Psi}_\epsilon$ stands for the solution of the free linear wave equation with wave speed $1/\epsilon$ and initial data $\Psi_{0,\epsilon}$ and $\Psi_{1,\epsilon}$. Firstly, we establish a bound for

$$|\mathrm{S}_\epsilon(t)| \le \|q\|_{L^1([0,\infty))}\|\sigma_{1,\epsilon} * \rho_\epsilon\|^2_{L^\infty(0,t;L^2(\mathbb{R}^3))}.$$

However, Lemma 2.4.7 yields

$$\|\sigma_{1,\epsilon} * \rho_\epsilon\|_{L^2(\mathbb{R}^3)} = C_d^2 \left\|\frac{\theta_\epsilon}{|\cdot|^2} * \delta_\epsilon * \rho_\epsilon\right\|_{L^2(\mathbb{R}^3)} \le C\|\rho_\epsilon\|_{L^{6/5}(\mathbb{R}^3)}.$$

Let us set

$$\mathcal{E}_\epsilon^{\mathrm{kin}}(t) = \int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f_\epsilon(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

for the particle kinetic energy. Lemma 2.4.6 leads to

$$\|\rho_\epsilon\|_{L^{5/3}(\mathbb{R}^3)} \le C(2,3)\|f_\epsilon\|^{2/5}_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\Big(\mathcal{E}_\epsilon^{\mathrm{kin}}\Big)^{3/5} \tag{2.27}$$

The Hölder inequality allows us to estimate $\|\rho_\epsilon\|_{L^{6/5}(\mathbb{R}^3)} \le \|\rho_\epsilon\|^{7/12}_{L^1(\mathbb{R}^3)}\|\rho_\epsilon\|^{5/12}_{L^{5/3}(\mathbb{R}^3)}$. Combining these inequalities, we arrive at

$$\|\sigma_{1,\epsilon} * \rho_\epsilon\|_{L^2(\mathbb{R}^3)} \le C\Big(\mathcal{E}_\epsilon^{\mathrm{kin}}\Big)^{1/4}, \tag{2.28}$$

for a certain constant $C > 0$, which does not depend on ϵ. Therefore, we obtain

$$|\mathrm{S}_\epsilon(t)| \le C^2\|q\|_{L^1([0,\infty))}\|\mathcal{E}_\epsilon^{\mathrm{kin}}\|^{1/2}_{L^\infty([0,t])}.$$

Secondly, we estimate the term involving $\Phi_{0,\epsilon}$:

$$\mathrm{T}_\epsilon(t) = \int_{\mathbb{R}^d\times\mathbb{R}^N} (\rho_\epsilon * \sigma_{1,\epsilon})(t,x)\widetilde{\Psi}_\epsilon(t,x,y)\sigma_2(y)\,\mathrm{d}y$$

is dominated by

$$\|\sigma_{1,\epsilon} * \rho_\epsilon\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}\|\widetilde{\Psi}_\epsilon\|_{L^\infty(\mathbb{R}_+;L^2(\mathbb{R}^d;L^{2n/(n-2)}(\mathbb{R}^n)))}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}.$$

Using (2.21) and (2.28), we get

$$|\mathrm{T}_\epsilon(t)| \le C'\Big(\mathcal{E}_\epsilon^{\mathrm{kin}}(t)\Big)^{1/4}\Big(\mathcal{E}_{0,\epsilon}^{\mathrm{vib}}\Big)^{1/2}$$

where the constant $C' > 0$ does not depend on ϵ. It remains to discuss how (**H7**)–(**H8**) implies a uniform estimate on the initial state. Note that $\mathrm{S}_\epsilon(0) = 0$. Hence, by using (**H8**), we are led to

$$\mathcal{E}^{\rm vib}_{0,\epsilon} + \frac{1}{2}\mathcal{E}^{\rm kin}_{\epsilon}(0) \leq \mathcal{E}_\epsilon(0) + |\mathrm{T}_\epsilon(0)| \leq \bar{\mathcal{E}}_0 + C'\Big(\mathcal{E}^{\rm kin}_{\epsilon}(0)\Big)^{1/4}\Big(\mathcal{E}^{\rm vib}_{0,\epsilon}\Big)^{1/2}.$$

It allows us to infer

$$\sup_{0<\epsilon<1} \mathcal{E}^{\rm kin}_{\epsilon}(0) = \bar{\mathcal{E}}^{\rm kin}_0 < \infty, \qquad \sup_{0<\epsilon<1} \mathcal{E}^{\rm vib}_{0,\epsilon} = \bar{\mathcal{E}}^{\rm vib}_0 < \infty.$$

Coming back to the energy conservation, with(**H7**)–(**H8**) together with the estimates on T_ϵ and S_ϵ, we deduce that

$$\frac{1}{2}\mathcal{E}^{\rm kin}_{\epsilon}(t) \leq \bar{\mathcal{E}}_0 + C^2\|q\|_{L^1([0,\infty))}\|\mathcal{E}^{\rm kin}_{\epsilon}\|^{1/2}_{L^\infty([0,t])} + C'\Big(\mathcal{E}^{\rm kin}_{\epsilon}(t)\Big)^{1/4}\Big(\bar{\mathcal{E}}^{\rm vib}_{0,\epsilon}\Big)^{1/2},$$

holds, which, in turn, establishes the bound

$$\sup_{0<\epsilon<1,\ t\geq 0} \mathcal{E}^{\rm kin}_{\epsilon}(t) = \bar{\mathcal{E}}^{\rm kin} < \infty.$$

Going back to the interpolation inequalities, it follows that ρ_ϵ is bounded in $L^\infty(0,\infty; L^1 \cap L^{5/3}(\mathbb{R}^3))$.

Step 2. Passing to the limit.

The kinetic equation can be rewritten

$$\partial_t f_\epsilon + v\cdot\nabla_x f_\epsilon - \nabla_x\Big(V + \Phi_{0,\epsilon} - \frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)\Big)\cdot\nabla_v f_\epsilon = 0.$$

We start by establishing that $\nabla_v f_\epsilon \cdot \nabla_x \Phi_{0,\epsilon} = \nabla_v \cdot (f_\epsilon \nabla_x \Phi_{0,\epsilon})$ converges to 0 at least in the sense of distributions.

Lemma 2.4.9 *Let $\chi \in C^\infty_c([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$. Then, we have*

$$\lim_{\epsilon\to 0}\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \nabla_x \Phi_{0,\epsilon}\chi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Proof. It is convenient to split

$$\begin{aligned}\Phi_{0,\epsilon}(t,x) &= \int_{\mathbb{R}^n} \sigma_2(y) C_3 \frac{\theta_\epsilon}{|\cdot|^2} * \delta_\epsilon * \widetilde{\Psi}_\epsilon(t,x,y)\,\mathrm{d}y \\ &= \Phi^{\rm main}_{0,\epsilon}(t,x) + \Phi^{\rm rem}_{0,\epsilon}(t,x)\end{aligned}$$

with

$$\Phi^{\rm main}_{0,\epsilon}(t,x) = \int_{\mathbb{R}^n} \sigma_2(y) C_3 \frac{1}{|\cdot|^2} * \delta_\epsilon * \widetilde{\Psi}_\epsilon(t,x,y)\,\mathrm{d}y,$$

$$\Phi^{\rm rem}_{0,\epsilon}(t,x) = \int_{\mathbb{R}^n} \sigma_2(y) C_3 \frac{\theta_\epsilon - 1}{|\cdot|^2} * \delta_\epsilon * \widetilde{\Psi}_\epsilon(t,x,y)\,\mathrm{d}y,$$

and we remind the reader that $\widetilde{\Psi}_\epsilon(t,x,y)$ is the solution of the free wave equation $(\epsilon\partial^2_{tt} - \Delta_y)\widetilde{\Psi}_\epsilon = 0$ with initial data $(\Psi_{0,\epsilon}, \Psi_{1,\epsilon})$. Accordingly, we are going to study the integral

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d\times\mathbb{R}^d} f_\epsilon \nabla_x \Phi_{0,\epsilon}\chi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad = \int_0^\infty \int_{\mathbb{R}^d} R_\epsilon(t,x)(\nabla_x\Phi^{\mathrm{main}}_{0,\epsilon} + \nabla_x\Phi^{\mathrm{rem}}_{0,\epsilon})(t,x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

with

$$R_\epsilon(t,x) = \int_{\mathbb{R}^d} f_\epsilon\chi(t,x,v)\,\mathrm{d}v$$

where χ is a given trial function, supported in $\{0 \le t \le M,\ |x| \le M,\ |v| \le M\}$ for some $0 < M < \infty$.

We observe that

$$\nabla_x\Big(\frac{\theta_\epsilon - 1}{|\cdot|^2} * g\Big) = \Big(\frac{\nabla_x\theta_\epsilon}{|\cdot|^2} - 2(\theta_\epsilon - 1)\frac{\cdot}{|\cdot|^4}\Big) * g.$$

Thus, by using (2.26) with $d = 3$ and $p = 2$, we are led to

$$|\nabla_x\Phi^{\mathrm{rem}}_{0,\epsilon}(t,x)| \le C\epsilon^{3/4}\left(\int_{\mathbb{R}^d}\left|\left(\delta_\epsilon * \int_{\mathbb{R}^n}\sigma_2(y)\widetilde{\Psi}_\epsilon(t,\cdot,y)\,\mathrm{d}y\right)(x')\right|^2\mathrm{d}x'\right)^{1/2}.$$

However, by (2.21) we have

$$\begin{aligned}&\left\|\delta_\epsilon * \int_{\mathbb{R}^n}\widetilde{\Psi}_\epsilon\sigma_2(y)\,\mathrm{d}y\right\|_{L^\infty([0,\infty);L^2(\mathbb{R}^3))}\\ &\qquad\le \|\delta_\epsilon\|_{L^1(\mathbb{R}^3)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sup_{t\ge 0}\left(\int_{\mathbb{R}^d}\|\widetilde{\Psi}_\epsilon(t,x,\cdot)\|^2_{L^{2n/(n-2)}(\mathbb{R}^n)}\,\mathrm{d}x\right)^{1/2}\\ &\qquad\le C\|\sigma_2\|_{L^{(n+2)/2n}(\mathbb{R}^n)}\left(\bar{\mathcal{E}}^{\mathrm{vib}}_0\right)^{1/2}.\end{aligned}$$

It implies that $\nabla_x\Phi^{\mathrm{rem}}_{0,\epsilon}(t,x)$ converges uniformly on $(0,\infty)\times\mathbb{R}^d$ to 0. Since R_ϵ is clearly bounded in $L^1((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$, we conclude that

$$\int_0^\infty\int_{\mathbb{R}^d} R_\epsilon\nabla_x\Phi^{\mathrm{rem}}_{0,\epsilon}\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[\epsilon\to 0]{} 0.$$

We need a more refined estimate to deal with the leading term $\Phi^{\mathrm{main}}_{0,\epsilon}$. We begin with

$$\begin{aligned}&\left|\int_0^\infty\int_{\mathbb{R}^d} R_\epsilon\nabla_x\Phi^{\mathrm{main}}_{0,\epsilon}\,\mathrm{d}x\,\mathrm{d}t\right|\\ &\qquad\le\left(\int_{\mathbb{R}^d}\left(\int_0^\infty|R_\epsilon|^{p'}\,\mathrm{d}t\right)^{2/p'}\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^d}\left(\int_0^\infty|\nabla_x\Phi^{\mathrm{main}}_{0,\epsilon}|^p\,\mathrm{d}t\right)^{2/p}\mathrm{d}x\right)^{1/2}.\end{aligned}$$

We realize that the components of $\nabla_x\Phi^{\mathrm{main}}_{0,\epsilon}$ are given by the solutions $\Upsilon_{j,\epsilon}$ of the wave equation

$$(\epsilon\partial^2_t - \Delta_y)\Upsilon_{j,\epsilon} = 0$$

with data

$$\Upsilon_{j,\epsilon}(0,x,y) = \partial_{x_j}\frac{C_3}{|\cdot|^2} * \delta_\epsilon * \Psi_{0,\epsilon}(x,y), \qquad \partial_t\Upsilon_{j,\epsilon}(0,x,y) = \partial_{x_j}\frac{C_3}{|\cdot|^2} * \delta_\epsilon * \Psi_{1,\epsilon}(x,y),$$

and the space variable $x \in \mathbb{R}^3$ has only the role of a parameter. It satisfies the following Strichartz estimate

$$\frac{1}{\epsilon^{1/(2p)}}\left(\int_0^\infty \left(\int_{\mathbb{R}^n} |\Upsilon_\epsilon(t,x,y)|^q \,\mathrm{d}y\right)^{p/q} \mathrm{d}t\right)^{1/p} \leq C\sqrt{\mathscr{E}^{\mathrm{vib}}_{1,\epsilon}(x)}$$

where

$$\mathscr{E}^{\mathrm{vib}}_{1,\epsilon}(x) = \epsilon\int_{\mathbb{R}^n} |\partial_t \Upsilon_\epsilon(0,x,y)|^2 \,\mathrm{d}y + \int_{\mathbb{R}^n} |\nabla_y \Upsilon_\epsilon(0,x,y)|^2 \,\mathrm{d}y$$

(for admissible exponents as detailed above). The Fourier transform of $x \mapsto \nabla_x \frac{C_3}{|x|^2}$ is $\frac{\xi}{|\xi|}$, see [62, Th. 5.9], which implies that the convolution operator $g \mapsto \nabla_x \frac{C_3}{|x|^2} * g$, is an isometry from $L^2(\mathbb{R}^3)$ to $(L^2(\mathbb{R}^3))^3$. Furthermore, we have $\|\delta_\epsilon * g\|_{L^2(\mathbb{R}^3)} \leq \|\delta_\epsilon\|_{L^1(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)} = \|g\|_{L^2(\mathbb{R}^3)}$. It follows that

$$\|\nabla_y \Upsilon_\epsilon(0)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^n_y)} \leq \|\nabla_y \Psi_{0,\epsilon}\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^n_y)}, \qquad \|\partial_t \Upsilon_\epsilon(0)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^n_y)} \leq \|\Psi_{1,\epsilon}\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^n_y)}.$$

Strichartz' estimate then leads to

$$\left(\int_{\mathbb{R}^d}\left(\int_0^\infty |\nabla_x \Phi^{\mathrm{main}}_{0,\epsilon}|^p \,\mathrm{d}t\right)^{2/p} \mathrm{d}x\right)^{1/2} \leq C\epsilon^{1/(2p)}\sqrt{\mathscr{E}^{\mathrm{vib}}_{0,\epsilon}} \leq C\epsilon^{1/(2p)}\sqrt{\bar{\mathscr{E}}^{\mathrm{vib}}_0}.$$

Since f_ϵ is bounded in $L^\infty(0,\infty;L^p(\mathbb{R}^d\times\mathbb{R}^d))$ for all $1 \leq p \leq \infty$, and χ is bounded and compactly supported we conclude that

$$\int_0^\infty\int_{\mathbb{R}^d} R_\epsilon \nabla_x \Phi^{\mathrm{main}}_{0,\epsilon} \,\mathrm{d}x\,\mathrm{d}t \xrightarrow[\epsilon\to 0]{} 0.$$

(Note that the same argument can be applied to show that $\nabla_x \Phi^{\mathrm{rem}}_{0,\epsilon}$ vanishes faster than what has been obtained with the mere energy estimate.) ■

Next, we study the non linear acceleration term. Let us set

$$\widetilde{\rho}_\epsilon(t,x) = \delta_\epsilon * \delta_\epsilon * \int_0^{t/\sqrt{\epsilon}} \rho_\epsilon(t - s\sqrt{\epsilon}, x)\; q(s)\,\mathrm{d}s.$$

It is clear, with Lemma 2.4.4, that $\widetilde{\rho}_\epsilon$ inherits from ρ_ϵ the uniform estimate $L^\infty(0,\infty;L^1\cap L^{5/3}(\mathbb{R}^3))$. We also denote $E(x) = \frac{1}{4\pi}\frac{1}{|x|}$, the elementary solution of the operator $-\Delta_x$ in $\mathbb{R}^3$. Note that $\nabla_x E(x) = -\frac{x}{4\pi|x|^3}$. Bearing in mind Lemma 2.4.8, the self–consistent field can be split as follows

$$\frac{1}{\epsilon}\nabla_x \mathcal{L}_\epsilon(f_\epsilon)(t,x) = \left[\nabla_x\Big(\frac{C_3\theta_\epsilon}{|\cdot|^2} * \frac{C_3\theta_\epsilon}{|\cdot|^2}\Big) - \nabla_x E\right] * \widetilde{\rho}_\epsilon(t,x) + \nabla_x E * \widetilde{\rho}_\epsilon(t,x). \tag{2.29}$$

In the right hand side, the L^r norm of the first term is dominated by $\|\widetilde{\rho}_\epsilon\|_{L^\infty([0,\infty;L^1(\mathbb{R}^3))}\Big\|\big[\ldots\big]\Big\|_{L^r(\mathbb{R}^3)}$, hence, owing to Lemma Lemma 2.4.8 it tends to 0 as $\epsilon \to 0$ in $L^\infty(0,\infty;L^r(\mathbb{R}^3))$ for any $3/2 < r < \infty$. Next, Lemma 2.4.7 tells us that

$$\nabla_x E * \widetilde{\rho}_\epsilon \text{ is bounded in } L^\infty(0,\infty;L^{15/4}(\mathbb{R}^3)).$$

Therefore, adapting the reasoning made in the previous sections, we deduce that we can extract a subsequence, such that, for any trial function $\chi \in L^{p'}(\mathbb{R}^3\times\mathbb{R}^3)$, $1/p' + 1/p = 1$,

$1 < p < \infty$,

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3} f_\epsilon(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)\chi(x,v)\,\mathrm{d}v\,\mathrm{d}x$$

holds uniformly on $[0,T]$, for any $0 \leq T < \infty$. Since the uniform estimate on the kinetic energy imply the tightness of f_ϵ with respect to the velocity variable, we also have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} \rho_\epsilon(t,x,v)\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \rho(t,x)\zeta(x)\,\mathrm{d}x, \qquad \rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v,$$

uniformly on $[0,T]$, for any $0 \leq T < \infty$ and any $\zeta \in L^q(\mathbb{R}^3)$, $q \geq 5/2$ or $\zeta \in C_0(\mathbb{R}^3)$. Clearly, for any $\zeta \in C_c^\infty(\mathbb{R}^3)$, $\delta_\epsilon * \delta_\epsilon * \zeta$ converges to ζ in $L^q(\mathbb{R}^3)$, $5/2 \leq q < \infty$, and in $C_0(\mathbb{R}^3)$. Therefore

$$\int_{\mathbb{R}^3} (\delta_\epsilon * \delta_\epsilon * \rho_\epsilon)(t,x)\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \rho_\epsilon(t,x)\ (\delta_\epsilon * \delta_\epsilon * \zeta)(x)\,\mathrm{d}x \xrightarrow[\epsilon\to 0]{} \kappa\ \int_{\mathbb{R}^3} \rho(t,x)\ \zeta(x)\,\mathrm{d}x$$

uniformly in $[0,T]$. Then, we look at the difference

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \widetilde{\rho}_\epsilon(t,x)\zeta(x)\,\mathrm{d}x - \kappa\int_{\mathbb{R}^3} \rho(t,x)\zeta(x)\,\mathrm{d}x\right| \\
&\quad \leq \int_0^{t/\sqrt{\epsilon}} \left|\int_{\mathbb{R}^3} (\delta_\epsilon * \delta_\epsilon * \rho_\epsilon)(t-\sqrt{\epsilon}s,x)\zeta(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} \rho(t-\sqrt{\epsilon}s,x)\zeta(x)\,\mathrm{d}x\right|\ |q(s)|\,\mathrm{d}s \\
&\qquad + \int_0^{t/\sqrt{\epsilon}} \left|\int_{\mathbb{R}^3} \rho(t-\sqrt{\epsilon}s,x)\zeta(x)\,\mathrm{d}x - \int_{\mathbb{R}^3} \rho(t,x)\zeta(x)\,\mathrm{d}x\right|\ |q(s)|\,\mathrm{d}s \\
&\qquad + \int_{t/\sqrt{\epsilon}}^\infty |q(s)|\,\mathrm{d}s\ \left|\int_{\mathbb{R}^3} \rho(t,x)\zeta(x)\,\mathrm{d}x\right|.
\end{aligned}$$

Let us denote by $\mathrm{I}_\epsilon(t)$, $\mathrm{II}_\epsilon(t)$ and $\mathrm{III}_\epsilon(t)$ the three integrals in the right hand side. By using Lemma 2.4.4 and the available estimates, we obtain, for any $0 \leq t \leq T < \infty$

$$|\mathrm{I}_\epsilon(t)| \leq \|q\|_{L^1([0,\infty))} \sup_{0\leq u\leq T}\left|\int_{\mathbb{R}^3} \Big(\delta_\epsilon * \delta_\epsilon * \rho_\epsilon - \rho\Big)(u,x)\zeta(x)\,\mathrm{d}x\right| \xrightarrow[\epsilon\to 0]{} 0,$$

while a direct application of the Lebesgue theorem shows that, for any $0 < t \leq T < \infty$

$$\lim_{\epsilon\to 0}\mathrm{II}_\epsilon(t) = 0 = \lim_{\epsilon\to 0}\mathrm{III}_\epsilon(t).$$

Therefore, for any $\zeta \in L^q(\mathbb{R}^3)$, $5/2 \leq q < \infty$ and any $\zeta \in C_0(\mathbb{R}^3)$,

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} \widetilde{\rho}_\epsilon(t,x)\zeta(x)\,\mathrm{d}x = \kappa\int_{\mathbb{R}^3} \rho(t,x)\zeta(x)\,\mathrm{d}x$$

holds for a. e. $t \in (0,T)$, with the domination

$$\left|\int_{\mathbb{R}^3} \widetilde{\rho}_\epsilon(t,x)\zeta(x)\,\mathrm{d}x\right| \leq \|\zeta\|_{L^{p'}(\mathbb{R}^3)} \sup_{\epsilon>0,\ 0\leq t\leq T} \|\rho_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^3)},$$

for any $1 \leq p \leq 5/3$.

In oder to justify that the limit f is a solution of the Vlasov–Poisson equation, the only difficulty relies on the treatment of the non linear acceleration term:

$$\mathrm{NL}_\epsilon(\chi) = \int_0^\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} f_\epsilon \nabla_x \frac{1}{\epsilon}\mathcal{L}_\epsilon(f_\epsilon)\cdot\nabla_v\chi\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t$$

where χ is a trial function in $\chi \in C_c^\infty([0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. Bearing in mind (2.29), it is convenient to rewrite

$$\mathrm{NL}_\epsilon(\chi) = \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f_\epsilon \nabla_v \chi \,\mathrm{d}v \right) \cdot \nabla_x E * \widetilde{\rho}_\epsilon \,\mathrm{d}x\,\mathrm{d}t + \mathscr{R}_\epsilon, \qquad \lim_{\epsilon\to 0} \mathscr{R}_\epsilon = 0.$$

Lemma 2.4.7 implies that $\nabla_x E * \widetilde{\rho}_\epsilon$ is bounded in $L^\infty(0,T; L^{15/4}(\mathbb{R}^3))$. For $\mu > 0$, we introduce the cut–off function $\widetilde{\theta}_\mu(x) = \theta(x/\mu)$. Then we split

$$\nabla_x E * \widetilde{\rho}_\epsilon(t,x) = \int_{\mathbb{R}^3} \widetilde{\theta}_\mu(x-y) \frac{x-y}{4\pi|x-y|^3} \widetilde{\rho}_\epsilon(t,y)\,\mathrm{d}y + \int_{\mathbb{R}^3} \Big(1 - \widetilde{\theta}_\mu(x-y)\Big) \frac{x-y}{4\pi|x-y|^3} \widetilde{\rho}_\epsilon(t,y)\,\mathrm{d}y.$$

The first term in the right hand side can be made arbitrarily small in L^p norm, $1 \le p \le 5/3$, uniformly with respect to ϵ, since it can be dominated by

$$\left\| \int_{|x-y|\le 2\mu} \frac{x-y}{4\pi|x-y|^3} \widetilde{\rho}_\epsilon(t,y)\,\mathrm{d}y \right\|_{L^p(\mathbb{R}^3)} \le \|\widetilde{\rho}_\epsilon(t,\cdot)\|_{L^p(\mathbb{R}^3)} \int_{|x-y|\le 2\mu} \frac{\mathrm{d}y}{4\pi|x-y|^2} \le C\ \mu.$$

In the second term, for fixed $x \in \mathbb{R}^3$ and μ, $y \mapsto \Big(1 - \widetilde{\theta}_\mu(x-y)\Big) \frac{x-y}{4\pi|x-y|^3|} \mathbf{1}_{|x-y|\ge\mu}$ is a continuous function which vanishes as $|y| \to \infty$, so that, for any $t > 0$,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^3} \Big(1 - \widetilde{\theta}_\mu(x-y)\Big) \frac{x-y}{4\pi|x-y|^3} \widetilde{\rho}_\epsilon(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^3} \Big(1 - \widetilde{\theta}_\mu(x-y)\Big) \frac{x-y}{4\pi|x-y|^3} \rho(t,y)\,\mathrm{d}y.$$

By standard arguments of integration theory (see for instance [47, Th. 7.61]), we deduce that (a suitable subsequence of) $\nabla_x E * \widetilde{\rho}_\epsilon$ converges to $\nabla_x E * \rho$ a. e. and strongly in $L^p_{\mathrm{loc}}((0,T) \times \mathbb{R}^3)$, for any $1 \le p < 15/4$. On the other hand, $\int_{\mathbb{R}^3} f_\epsilon \nabla_v \chi \,\mathrm{d}v$ is compactly supported and converges to $\int_{\mathbb{R}^3} f_\epsilon \nabla_v \chi \,\mathrm{d}v$ weakly in any $L^q((0,T) \times \mathbb{R}^3)$. (In fact this convergence, as well as $\rho_\epsilon \to \rho$ can be shown to hold strongly, by applying average lemma techniques, see [32, Th. 5].) We conclude that

$$\lim_{\epsilon\to 0} \mathrm{NL}_\epsilon(\chi) = \int_0^\infty \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} f \nabla_v \chi \,\mathrm{d}v \right) \cdot \nabla_x E * \rho \,\mathrm{d}x\,\mathrm{d}t.$$

It ends the proof of Theorem 2.4.2. ■

Chapitre 3

Le deuxième modèle obtenu en ajoutant un terme de Fokker-Planck

Dans cet article écrit en collaboration avec Ricardo Alonzo et Thierry Goudon, nous ajoutons un terme de Fokker-Planck à l'équation d'évolution de la fonction de distribution des particules et nous plaçons dans le cas où V est un potentiel de confinement. Nous étudions d'abord le régime asymptotique de diffusion pour ce nouveau modèle en le combinant avec le premier régime asymptotique étudié dans le premier chapitre. Lorsque la vitesse de propagation des ondes est suffisament grande, nous montrons qu'il existe un unique état d'équilibre au système. A l'aide de méthodes d'hypocoercivité que l'on adapte de [34], nous montrons que les solutions de l'équation convergent toutes vers cet équilibre.

3.1 Introduction

This work is concerned with the long–time behavior of the solution of the Vlasov equation

$$\partial_t F + v\cdot\nabla_x F - \nabla_x(V+\Phi)\cdot\nabla_v F = \gamma\nabla_v\cdot(vF+\nabla_v F), \qquad t\geq 0,\ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d, \tag{3.1}$$

where Φ is self–consistently defined by the relations

$$\begin{cases} \Phi(t,x) = \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-y)\sigma_2(z)\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, & t\geq 0,\ x\in\mathbb{R}^d,\\ \Big(\partial_{tt}^2\Psi - c^2\Delta_z\Psi\Big)(t,x,z) = -\sigma_2(z)\ \displaystyle\int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y, & t\geq 0,\ x\in\mathbb{R}^d,\ z\in\mathbb{R}^n,\\ \text{with } \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} F(t,x,v)\,\mathrm{d}v. \end{cases} \tag{3.2}$$

The system is complemented with the initial data

$$F(0,x,v) = F_0(x,v), \qquad \Psi(0,x,z) = \Psi_0(x,z), \qquad \partial_t \Psi(0,x,z) = \Psi_1(x,z). \tag{3.3}$$

The parameters of the problem are set as follows

- $c > 0$,
- $\sigma_1 : \mathbb{R}^d \to [0,\infty)$ and $\sigma_2 : \mathbb{R}^n \to [0,\infty)$ are radially symmetric C^∞ compactly supported functions,
- $V : \mathbb{R}^d \to \mathbb{R}$ is an external *confining* potential:

$$V \in C^0 \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^d), \qquad \lim_{|x|\to\infty} V(x) = \infty.$$

We will make the technical assumptions precise later on. A crucial role in the analysis will be played by the following entropy dissipation property

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(F\frac{v^2}{2} + F(V+\Phi) + F\ln(F)\Big)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} (|\partial_t\Psi|^2 + c^2|\nabla_z\Psi|^2)\,\mathrm{d}z\,\mathrm{d}x\Big\} \\ = -\gamma\int_{\mathbb{R}^d\times\mathbb{R}^d} |2\nabla_v\sqrt{F} + v\sqrt{F}|^2\,\mathrm{d}v\,\mathrm{d}x \le 0.\end{aligned} \tag{3.4}$$

The investigation of this problem is motivated by the work of S. De Bièvre and L. Bruneau [17] where a related model was introduced to describe the evolution of a single particle interacting with its environment. In [17] the particle is classically described by the pair position/velocity $(q(t), \dot q(t))$, and the dynamics is governed by

$$\begin{cases} \ddot q(t) = -\nabla V(q(t)) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q(t)-y)\ \sigma_2(z)\ \nabla_x\Psi(t,y,z)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,z) - c^2\Delta_z\Psi(t,x,z) = -\sigma_2(z)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ z\in\mathbb{R}^n. \end{cases} \tag{3.5}$$

Such single particle description can be retrieved by setting $F(t,x,v) = \delta(x = q(t)) \otimes \delta(v = \dot q(t))$ in (3.1), with $\gamma = 0$. The dynamics can be thought of as if membranes continuously distributed transversely to the direction of the particle's motion — $z \in \mathbb{R}^n$ being transerse to $x \in \mathbb{R}^d$ — were activated by the passage of the particle, see Fig. 1 in [17]. The evolution of the system is, therefore, driven by energy exchanges between the particle and the membranes. We remark that the coupling between the particle and the membranes is embodied into the *product* $\sigma_1(x)\sigma_2(z)$, which appears symmetrically in the two equations of (3.5). This is crucial to establish Hamiltonian properties of (3.5) and its counterpart for the kinetic model, namely relation (3.4). The system is presented as a "dynamical Lorentz gas" and one is interested in asymptotic properties of the dynamics. This question has been further investigated in a series of papers by S. De Bièvre and his collaborators [1, 26, 27, 28, 59], that contains both theoretical results and convincing numerical experiments. On the one hand, the system has certain dissipative features: under certain circumstances (roughly speaking, $n = 3$ and c large enough) the particle energy can be dissipated in the membranes, and the environment behaves like a friction force on the particle. In particular, when V is a confining potential with a (non–degenerate) minimum at q_0, then the particle stops at the location q_0 as time goes to ∞, see [17, Section 5, Theorem 4]. On the other hand, in [1, 27] an approximated model is proposed, together with an interpretation of the dynamics in terms of random walk. This simplified framework permits to justify the approach to thermal equilibrium: the particle's momentum distribution is driven to a Maxwell-Boltzmann distribution.

We wish to revisit these questions in the framework of kinetic equations, where the description by the position/velocity pair is replaced by (3.1) when considering a distribution of particles in phase space $F(t, x, v) \geq 0$. More precisely, in the case $\gamma = 0$, $\int_A F(t, x, v)\,\mathrm{d}v\,\mathrm{d}x$ can be interpreted as the probability $\mathbb{P}\Big((q(t), \dot{q}(t)) \in A\Big)$ when the initial state of the particle is distributed according to F_0. The analysis of existence and uniqueness of weak solutions for the non linearly coupled problem (3.1)–(3.2), with $\gamma = 0$, was established in [**?**], where it was also shown that a certain physical regime drives the solutions of (3.1)–(3.2) to solutions of the attractive Vlasov–Poisson system. It is likely that this approach can be combined to the analysis of the smoothing effect of the Fokker–Planck operator in [12, 13] in order to investigate the well-posedness of the problem when $\gamma > 0$. We will not elaborate more on this issue in this paper and, instead, focus on the long–time behavior of the solutions of (3.1)–(3.2). We treat the question by adding a dissipative structure through the Fokker-Planck term $\gamma\nabla_v \cdot (vF + \nabla_v F)$, with $\gamma > 0$, in the right hand side of (3.1). It corresponds to consider a large set of particles governed by (3.5) where, in addition, we add a friction term $-\gamma\dot{q}(t)$ and a Brownian motion term which can be attributed to the positive temperature of the medium [17, 26]. We refer the reader to [80] for the analysis of the mean field regime that drives from the particles description to the Fokker-Planck equation. Although this term drastically simplifies the objectives of [17], since the model with $\gamma = 0$ is supposed to contain by itself friction/dissipation mechanisms, the dissipative model already leads to non trivial issues due to non-linear coupling of the interactions, and this first attempt on the PDE system (3.1)–(3.2) confirms the intuition that comes from the analysis of (3.5). Moreover, as a by-product, we are able to identify a family of stationary solutions of the system when $\gamma = 0$, which is less obvious than for the case of a single particle, and we establish that these solutions are linearly stable for the dissipationless model.

The paper is organized as follows. In Section 3.2, we will exhibit stationary solutions of (3.1)–(3.2). Having introduced the necessary notation, we give the statement of our main result, namely the convergence to equilibrium at exponential rate. As a preliminary step to understand the long–time behavior, it is convenient to discuss the so-called "diffusive scaling" for the problem (3.1)–(3.2). This is the object of Section 3.3. In Section 3.4, we investigate the large time behavior of the solutions. Our analysis relies on the assumption that the wave speed is sufficiently large. In this regime (3.1) appears, in some sense, as a perturbation of the linear Fokker–Planck equation with external potential V. In this context, the method recently presented in [34] based on hypocoercivity arguments becomes quite useful. We will follow such an approach where, roughly speaking, the goal is to define a suitable Lyapounov functional which combines the natural entropy of the problem and an additional inner product that allows us to control the hydrodynamic part of the solution. Furthermore, the solutions exhibited in Section 3.2 are also stationary solutions of the dissipationless model ($\gamma = 0$); we investigate their linearized stability in the Appendix.

3.2 Equilibrium states

We rewrite the Fokker-Planck operator, hereafter denoted by L, as follows

$$LF = \nabla_v \cdot (vF + \nabla_v F) = \nabla_v \cdot \Big(M \nabla_v \frac{F}{M} \Big), \qquad M(v) = (2\pi)^{-d/2} e^{-v^2/2}.$$

This form indicates the dissipative effect of this operator; in particular we have

$$\int_{\mathbb{R}^d} LF \frac{F}{M} \, \mathrm{d}v = - \int_{\mathbb{R}^d} M \Big| \nabla_v \Big(\frac{F}{M} \Big) \Big|^2 \mathrm{d}v,$$

which already shows that $\mathrm{Ker}(L) = \mathrm{Span}(M)$. We search for equilibrium solutions of (3.1)–(3.2), which means solutions independent of the time variable t, that make both the "transport part" and the "collisional part" of the equation vanish, namely we seek $\mathscr{M}_{\mathrm{eq}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, such that

$$\text{(a)} \quad L\mathscr{M}_{\mathrm{eq}} = 0, \qquad \text{(b)} \quad (v \cdot \nabla_x - \nabla_x (V + \Phi_{\mathrm{eq}}) \cdot \nabla_v) \mathscr{M}_{\mathrm{eq}} = 0.$$

Condition (b) is reached by any function depending on the total energy $v^2/2 + (V + \Phi_{\mathrm{eq}})(x)$, while, as said above, the kernel of L imposes a precise dependence with respect to the velocity variable. Combining (a) and (b), therefore, leads to

$$\mathscr{M}_{\mathrm{eq}}(x, v) = Z_{\mathrm{eq}} \ \exp\Big(- \frac{v^2}{2} - V(x) - \Phi_{\mathrm{eq}}(x) \Big).$$

In this formula, Z_{eq} is a normalizing factor. Indeed, (3.1) is mass preserving in the sense that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} F(t, x, v) \, \mathrm{d}v \, \mathrm{d}x = \int_{\mathbb{R}^d \times \mathbb{R}^d} F(0, x, v) \, \mathrm{d}v \, \mathrm{d}x \stackrel{\text{def}}{=} \mathfrak{m},$$

and therefore Z_{eq} is such that $\mathscr{M}_{\mathrm{eq}}$ has also mass $\mathfrak{m}$, which yields

$$Z_{\mathrm{eq}} = \mathfrak{m} \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} e^{-v^2/2 - V(x) - \Phi_{\mathrm{eq}}(x)} \, \mathrm{d}v \, \mathrm{d}x \Big)^{-1} = \frac{\mathfrak{m}}{(2\pi)^{d/2}} \Big(\int_{\mathbb{R}^d} e^{-V(x) - \Phi_{\mathrm{eq}}(x)} \, \mathrm{d}x \Big)^{-1}.$$

However, we should take into account the non linearity of the problem by revisiting the definition of the self–consistent potential in (3.2). Considering stationary solutions, (3.2) becomes

$$\begin{aligned}
&-c^2\Delta_z \Psi_{\mathrm{eq}}(x,z) = -\sigma_2(z)\ \sigma_1 * \rho_{\mathrm{eq}}(x),\\
&\rho_{\mathrm{eq}}(x) = \int_{\mathbb{R}^d} \mathscr{M}_{\mathrm{eq}}(x,v)\,\mathrm{d}v,\\
&\Phi_{\mathrm{eq}}(x) = \left(\sigma_1 * \int_{\mathbb{R}^n} \sigma_2(z)\Psi_{\mathrm{eq}}(\cdot,z)\,\mathrm{d}z\right)(x).
\end{aligned}$$

For further purposes, it is convenient to keep in mind the following notation

$$\mathscr{M}_{\mathrm{eq}}(x,v) = \rho_{\mathrm{eq}}(x)M(v), \qquad \rho_{\mathrm{eq}}(x) = (2\pi)^{d/2} Z_{\mathrm{eq}} e^{-(\Phi_{\mathrm{eq}}+V)(x)}.$$

For the stationary problem, the space variable x and the transverse variable z decouple. Let $\Upsilon : \mathbb{R}^n \to \mathbb{R}$ be the solution of

$$-\Delta_z \Upsilon = \sigma_2$$

(defined by the convolution of σ_2 by the fundamental solution of $(-\Delta)$ in $\mathbb{R}^n$). We obtain

$$\Psi_{\mathrm{eq}}(x,z) = -\frac{1}{c^2}\Upsilon(z)\ \sigma_1 * \rho_{\mathrm{eq}}(x).$$

It follows that the equilibrium potential satisfies

$$\Phi_{\mathrm{eq}}(x) = -\frac{\Lambda}{c^2}\ \Sigma * \rho_{\mathrm{eq}}(x)$$

where

$$\Lambda = \int_{\mathbb{R}^n} \sigma_2(z)\Upsilon(z)\,\mathrm{d}z, \qquad \Sigma = \sigma_1 * \sigma_1.$$

As far as $n \geq 3$, we justify the integration by parts that leads to

$$\Lambda = \int_{\mathbb{R}^n} |\nabla_z \Upsilon(z)|^2\,\mathrm{d}z \in (0,\infty).$$

Eventually, by combining the information, we are led to define the equilibrium potential as the solution of the nonlinear equation

$$\Phi_{\mathrm{eq}}(x) = -\frac{(2\pi)^{d/2}\Lambda}{c^2}\ Z_{\mathrm{eq}}\ \int_{\mathbb{R}^d} \Sigma(x-y)e^{-V(y)-\Phi_{\mathrm{eq}}(y)}\,\mathrm{d}y. \tag{3.6}$$

This discussion motivates the introduction of the following mapping

$$\mathscr{T} : \Phi \mapsto -\Lambda Z[\Phi]\ \int_{\mathbb{R}^d} \Sigma(x-y)e^{-V(y)-\Phi(y)}\,\mathrm{d}y, \qquad Z[\Phi] = \frac{\mathfrak{m}}{(2\pi)^{d/2}}\left(\int_{\mathbb{R}^d} e^{-V(x)-\Phi(x)}\,\mathrm{d}x\right)^{-1},$$

and to define equilibrium states as fixed point of $\frac{1}{c^2}\mathscr{T}$. Before stating our first result, let us collect here the confining assumptions on the external potential:

$$e^{-V} \in L^1(\mathbb{R}^d). \tag{\textbf{A1}}$$

$$\liminf_{|x|\to\infty}\left(\left|\nabla_x V(x)\right|^2 - 2\Delta_x V(x)\right) > 0. \tag{\textbf{A2}}$$

$$\begin{aligned}&\text{There exists } c_1, c_2 > 0, \text{ and } 0 < c_3 < 1 \text{ such that}\\ &\Delta_x V \le c_1 + \frac{c_3}{2}\Big|\nabla_x V\Big|^2, \qquad \Big|D_x^2 V\Big| \le c_2\Big(1 + \Big|\nabla_x V\Big|\Big).\end{aligned} \tag{\textbf{A3}}$$

For the existence of equilibrium states, only (**A1**) will be useful; the other assumptions will be used for the analysis of the large time behavior.

Theorem 3.2.1 *Let $n \ge 3$. Assume* (**A1**). *There exists $c_0 > 0$ such that for any $c > c_0$, the mapping $\frac{1}{c^2}\mathscr{T}$ admits a unique fixed point $\Phi \in C^0 \cap L^\infty(\mathbb{R}^d)$. If $0 < c \le c_0$, $\frac{1}{c^2}\mathscr{T}$ admits at least one fixed point.*

Proof. Let ρ be a non negative function such that $\int_{\mathbb{R}^d} \rho \,\mathrm{d}x = \mathfrak{m}$. We also suppose that the product $\rho\, e^V$ belongs to L^∞. Then, $\hat\Phi : x \mapsto \hat\Phi(x) = -\frac{(2\pi)^{d/2}\Lambda}{c^2}\Sigma * \rho(x)$ is continuous and satisfies

$$0 \ge \hat\Phi(x) \ge -\frac{(2\pi)^{d/2}\Lambda}{c^2}\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m} \stackrel{\text{def}}{=} -\frac{\kappa}{c^2}.$$

It follows that, on the one hand

$$0 \le e^{-V} \le e^{-\hat\Phi - V} \le e^{\kappa/c^2}\ e^{-V}$$

and, on the other hand

$$\frac{\mathfrak{m}}{(2\pi)^{d/2}}\ e^{-\kappa/c^2}\left(\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x\right)^{-1} \le Z[\hat\Phi] \le \frac{\mathfrak{m}}{(2\pi)^{d/2}}\ \left(\int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x\right)^{-1}. \tag{3.7}$$

By applying this reasoning to $\rho = (2\pi)^{d/2}Z[\Phi]e^{-V-\Phi}$, we conclude that $\frac{1}{c^2}\mathscr{T}$ leaves invariant the set

$$\mathscr{C} = \Big\{\Phi \in C^0(\mathbb{R}^d),\ -\kappa/c^2 \le \Phi \le 0\Big\}.$$

Furthermore, for $\Phi, \Phi' \in \mathscr{C}$, we obtain (with obvious notation)

$$\Big|\mathscr{T}(\Phi)(x) - \mathscr{T}(\Phi')(x)\Big| = \Big| - \Lambda\Sigma * (\rho - \rho')(x)\Big| \le \Lambda\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\|\rho - \rho'\|_{L^1}$$

with

$$\begin{aligned}\Big|\rho(x) - \rho'(x)\Big| &= (2\pi)^{d/2}e^{-V(x)}\Big|Z[\Phi]e^{-\Phi(x)} - Z[\Phi']e^{-\Phi'(x)}\Big|\\ &\le (2\pi)^{d/2}e^{-V(x)}\Big(e^{-\Phi(x)}\Big|Z[\Phi] - Z[\Phi']\Big| + Z[\Phi']\Big|e^{-\Phi(x)} - e^{-\Phi'(x)}\Big|\Big).\end{aligned}$$

Since the elements of $\mathscr{C}$ are bounded, we find

$$\Big|e^{-\Phi(x)} - e^{-\Phi'(x)}\Big| \le e^{\kappa/c^2}\Big|\Phi(x) - \Phi'(x)\Big| \le e^{\kappa/c^2}\|\Phi - \Phi'\|_{L^\infty}.$$

Similarly, by using (3.7), we obtain

$$\Big|Z[\Phi] - Z[\Phi']\Big| \le e^{\kappa/c^2}\frac{\mathfrak{m}}{(2\pi)^{d/2}\|e^{-V}\|_{L^1}}\|\Phi - \Phi'\|_{L^\infty}.$$

Gathering these estimates we conclude that

$$\Big|\mathscr{T}(\Phi)(x) - \mathscr{T}(\Phi')(x)\Big| \le 2\Lambda\|\Sigma\|_\infty\mathfrak{m}\ e^{\kappa/c^2}\ \|\Phi - \Phi'\|_{L^\infty}.$$

Therefore $\frac{1}{c^2}\mathscr{T}$ is Lipschitz, with a constant that tends to 0 as $c \to \infty$; we conclude by a direct application of the Banach Fixed Point Theorem, provided c is large enough. Furthermore, we can also remark that $\nabla_x \mathscr{T}(\Phi)$ is bounded uniformly for any $\Phi \in \mathscr{C}$. By virtue of the Ascoli Theorem, the mapping $\frac{1}{c^2}\mathscr{T}$ is therefore compact on $\mathscr{C}$. The Schauder Theorem proves that $\frac{1}{c^2}\mathscr{T}$ admits at least one fixed point in $\mathscr{C}$; however, uniqueness of the normalized equilibrium state is not guaranteed unless c is sufficiently large. ■

Note that the regularity of Φ_{eq}, and that of $\rho_{\text{eq}} = (2\pi)^{d/2} Z[\Phi_{\text{eq}}] e^{-(\Phi_{\text{eq}}+V)}$, is determined by the regularity of V and σ_1. Additionally observe that

$$0 \leq \rho_{\text{eq}} \leq \frac{\mathfrak{m} e^{\kappa/c^2}}{\|e^{-V}\|_{L^1}} e^{-V}.$$

Remark 3.2.2 *We can interpret the equilibrium states and the role of the smallness assumption on c in terms of the minimization of the following energy functional*

$$\mathscr{E}[\rho] = \int_{\mathbb{R}^d} \Big(\rho \ln(\rho) - \frac{\Lambda}{2c^2} \rho \Sigma * \rho + V\rho \Big) \, \mathrm{d}x$$

over integrable non negative functions with total mass $\mathfrak{m}$. Indeed, with the associated Euler-Lagrange equations we recover the definition of ρ_{eq} and Φ_{eq}. However, we observe that $\rho \mapsto \mathscr{E}[\rho]$ is strictly convex for c large enough, but due to the minus sign in front of the quadratic term, convexity might be lost for small c's.

Let us finish this section stating the main result of the paper which establishes the exponential trend to equilibrium, see Section 3.4.

Theorem 3.2.3 *Suppose $n = 3$, and let $\mathscr{E}_0, \mathfrak{m} > 0$ be fixed. We assume that the external potential satisfies* (**A1**), (**A2**), (**A3**). *Then, there exists $c_1 \geq c_0 > 0$ and $\kappa > 0$ such that, for any $c \geq c_1$, and any datum in* (3.3) *that fulfils the conditions*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} F_0 \, \mathrm{d}v \, \mathrm{d}x = \mathfrak{m}, \tag{\textbf{A4}}$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^3} \big(|\nabla_z (\Psi_0 - \Psi_{\text{eq}})|^2 + |\Psi_1|^2 \big) \, \mathrm{d}z \, \mathrm{d}x \leq \mathscr{E}_0, \tag{\textbf{A5}}$$

$$F_0 - \mathscr{M}_{\text{eq}} \in L^2\Big(\mathbb{R}^d \times \mathbb{R}^d; \frac{\mathrm{d}v \, \mathrm{d}x}{\mathscr{M}_{\text{eq}}(x,v)} \Big), \tag{\textbf{A6}}$$

$$\operatorname{supp}(\Psi_0 - \Psi_{\text{eq}}, \Psi_1) \subset \mathbb{R}^d \times B(0, R_{\text{I}}), \tag{\textbf{A7}}$$

we can find $M > 0$ such that the solution of (3.1)–(3.3) *satisfies*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|F(t,x,v) - \mathscr{M}_{\text{eq}}(x,v)|^2}{\mathscr{M}_{\text{eq}}(x,v)} \, \mathrm{d}v \, \mathrm{d}x \leq M e^{-\kappa t}.$$

We point out the fact that all the constants c_1, κ, M can be explicitly computed by means of the data $\sigma_1, \sigma_2, F_0, \Psi_0, \Psi_1$. In particular, M is proportional to

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|F_0 - \mathscr{M}_{\rm eq}|^2}{\mathscr{M}_{\rm eq}}\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^3} \Big(|\nabla_z(\Psi_0 - \Psi_{\rm eq})|^2 + |\Psi_1|^2\Big)\,\mathrm{d}z\,\mathrm{d}x,$$

which are the norms involved in (**A5**) and (**A6**).

Remark 3.2.4 *It is clear that the result can be extended to the full variety of collision operators considered in [34]; in particular it applies to, maybe less physical in this context, linear Boltzmann operators. Furthermore, as it will be clear within the proof, the regularity assumption on σ_1 and σ_2 can be relaxed and it is likely that the radial symmetry is not essential, but these assumptions stick to the framework introduced in [17].*

3.3 Diffusion asymptotics

As it is recalled in [34, Section 1.2], the intuition on the large time asymptotics can be motivated by investigating first a certain regime, where the PDEs system is rescaled by means of a relevant parameter $0 < \epsilon \ll 1$. Roughly speaking, we rescale the problem so that the Fokker–Planck term becomes stiff. It makes the relaxation effects strong. Since the flux of the equilibrium functions in $\mathrm{Ker}(L)$ vanishes, time and space scales should be appropriately rescaled in order to obtain a non trivial problem in the limit $\epsilon \to 0$. It can formally be understood through the change of variables $t \to \epsilon^2 t$, $x \to \epsilon x$. Such regimes are usually referred to as “diffusive regimes” in kinetic theory since one is led to a diffusion equation for the limiting macroscopic density. We shall start by introducing more precisely the scaling of the equations, by means of the physical parameters of the model. We will also explain how to rescale the coupling term in the wave equation. Next, we can guess the asymptotic behavior by expanding formally the solution as a power series of the parameter ϵ. Finally, we state and prove the convergence of the solutions as ϵ tends to 0. Entropy dissipation is a crucial ingredient of the proof.

Scaling of the equations

We follow the dimension analysis proposed in [25]. Let T, L be time and space units, respectively. In dimensional form the right hand side of the kinetic equation should be written

$$\frac{1}{\tau}\nabla_v \cdot (vF + \mathscr{V}^2\nabla_v F),$$

where the coefficient $\frac{1}{\tau}$ has the homogeneity of the inverse of time and $\mathscr{V}$ has the homogeneity of a velocity. We use $\mathscr{V}$ as a typical size for the velocity fluctuation (the thermal velocity in other contexts). The dimension of the particle distribution function is $\mathrm{L}^{-d}\mathscr{V}^{-d}$, so that $\int F\,\mathrm{d}v\,\mathrm{d}x$ is dimensionless. Considering the energy balance, $|\partial_t\Psi|^2\,\mathrm{d}y\,\mathrm{d}x$ and $c^2|\nabla_y\Psi|^2\,\mathrm{d}y\,\mathrm{d}x$

have the same dimension as $v^2 F \,\mathrm{d}v\,\mathrm{d}x$, that is the square of a velocity. With these remarks, we define dimensionless quantities, denoted by $\cdot'$, as follows

$$t = \mathrm{T}t', \qquad x = \mathrm{L}x', \qquad v = \mathscr{V} v',$$
$$\mathrm{L}^d \mathscr{V}^d F(t,x,v) = F'(t',x',v').$$

The external potential scales as $V(x) = \mathrm{U}V'(x')$, where U has the homogeneity of the square of the velocity. For the vibrating field, we set

$$y = \mathrm{L}y', \qquad \sqrt{\mathrm{L}^{d+n}}\,\Psi(t,x,z) = \mathrm{L}\,\Psi'(t',x',z').$$

Finally, the form functions are rescaled as follows

$$\sigma_1(x) = \varsigma_1\sigma_1'(x'), \qquad \sigma_2(z) = \varsigma_2\sigma_2'(z'),$$

with the suitable units for ς_1 and ς_2. In particular, we assume that the external potential and the self-consistent potential have the same order of magnitude, which leads to $\mathrm{U} = \varsigma_1\varsigma_2 L^{1+n/2+d/2}$ (note that the individual units of σ_1 and σ_2 does not really matter, the important quantity being their product). We can rewrite the PDE system in dimensionless form

$$\begin{aligned}
&\partial_t F + \frac{\mathscr{V}\mathrm{T}}{\mathrm{L}} v\cdot\nabla_x F - \frac{\mathrm{UT}}{\mathrm{L}\mathscr{V}}\nabla_x(V+\Phi)\cdot\nabla_v F = \frac{\mathrm{T}}{\tau} LF,\\
&\partial^2_{tt}\Psi - \left(\frac{c\mathrm{T}}{\mathrm{L}}\right)^2 \Delta_y\Psi(t,x,z) = -\frac{\mathrm{UT}^2}{\mathrm{L}^2}\sigma_2(z)\int_{\mathbb{R}^d\times\mathbb{R}^d}\sigma_1(x-y)F(t,y,v)\,\mathrm{d}v\,\mathrm{d}z,
\end{aligned}$$

where we get rid of the $\cdot'$ for simplicity of notation. The self–consistent potential is given by

$$\Phi(t,x) = \int_{\mathbb{R}^n\times\mathbb{R}^d}\sigma_2(y)\sigma_1(x-z)\Psi(t,z,y)\,\mathrm{d}z\,\mathrm{d}y.$$

We are interested in the regime where

$$\frac{\mathrm{T}}{\tau} = \frac{1}{\epsilon^2} = \frac{\mathrm{UT}^2}{\mathrm{L}^2}, \qquad \frac{\mathscr{V}\mathrm{T}}{\mathrm{L}} = \frac{1}{\epsilon} = \frac{c\mathrm{T}}{\mathrm{L}},$$

with $0 < \epsilon \ll 1$. Note that both the particles and the waves are "fast" compared to the velocity of observation, with speeds of the same order of magnitude. We arrive at

$$\begin{aligned}
&\partial_t F_\epsilon + \frac{1}{\epsilon}(v\cdot\nabla_x - \nabla_x(V+\Phi_\epsilon)\cdot\nabla_v)F_\epsilon = \frac{1}{\epsilon^2}LF_\epsilon,\\
&\partial^2_{tt}\Psi_\epsilon - \frac{1}{\epsilon^2}\Delta_z\Psi_\epsilon(t,x,z) = -\frac{1}{\epsilon^2}\sigma_2(z)\int_{\mathbb{R}^d\times\mathbb{R}^d}\sigma_1(x-y)F_\epsilon(t,y,v)\,\mathrm{d}v\,\mathrm{d}y,
\end{aligned} \tag{3.8}$$

with $\Phi_\epsilon(t,x) = \Big(\sigma_1 * \int_{\mathbb{R}^n}\sigma_2(z)\Psi_\epsilon(t,\cdot,z)\,\mathrm{d}z\Big)(x)$. The entropy dissipation (3.4) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Bigg(&\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(\frac{v^2}{2} + V + \Phi_\epsilon + \ln(F_\epsilon)\Big)F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\\
&+\frac{\epsilon^2}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\partial_t\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^d}|\nabla_z\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x\Bigg)\\
&= -\frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|2\nabla_v\sqrt{F_\epsilon} + v\sqrt{F_\epsilon}\Big|^2\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Formal asymptotic by Hilbert expansion

In order to guess the asymptotic behavior of the system for small ϵ's, we expand the solution as follows

$$F_\epsilon = F^{(0)} + \epsilon F^{(1)} + \epsilon^2 F^{(2)} + ...$$

and we plug this expansion into (3.8). We identify terms arising with the same exponent of ϵ:

a) ϵ^{-2} terms: we get $LF^{(0)} = 0$ which yields $F^{(0)}(t,x,v) = \rho(t,x)M(v)$,

b) ϵ^{-1} terms: the relation $LF^{(1)} = (v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)F^{(0)} = vM(\nabla_x\rho + \rho\nabla_x(V+\Phi))$ yields $F^{(1)}(t,x,v) = -vM(v)(\nabla_x\rho(t,x) + \rho\nabla_x(V+\Phi)(t,x)) + \rho^{(1)}(t,x)M(v)$,

c) ϵ^0 terms: we obtain $LF^{(2)} = \partial_t F^{(0)} + (v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)F^{(1)}$. Integrating with respect to the velocity variable and taking into account the expression for $F^{(1)}$ obtained in Step b) lead to

$$\partial_t\rho - \nabla_x\cdot(\nabla_x\rho + \rho\nabla_x(V+\Phi)) = 0. \tag{3.9}$$

(Note that the term $\rho^{(1)}(t,x)M(v) \in \mathrm{Ker}(L)$ does not contribute to the equation.)

The self consistent potential is determined by considering the leading terms in the wave equation. We arrive at the relation

$$\Phi(t,x) = -\Lambda\int_{\mathbb{R}^d}\Sigma(x-y)\rho(t,y)\,\mathrm{d}y. \tag{3.10}$$

Therefore, as $\epsilon \to 0$, we expect that $F_\epsilon(t,x,v)$ converges to $\rho(t,x)M(v)$, with ρ solution of the system (3.9)–(3.10).

Theorem 3.3.1 *We assume $n \geq 3$. We slightly strengthen the confining assumption (**A1**), by assuming $e^{-\nu V} \in L^1(\mathbb{R}^d)$ for some $\nu \in (0,1/2)$. Let us denote*

$$\begin{aligned}\mathscr{K}_0 = \sup_{\epsilon>0}\Big\{\int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon(0,x,v)\Big(1 + |\ln(F_\epsilon)(0,x,v)| + V(x) + \frac{v^2}{2}\Big)\,\mathrm{d}v\,\mathrm{d}x \\ +\frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\Psi_\epsilon(0,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x\Big\}\end{aligned} \tag{3.11}$$

which is assumed to be finite. Then, up to a subsequence, the solutions $F_\epsilon(t,x,v)$ of (3.8) converge as $\epsilon \to 0$ to $\rho(t,x)M(v)$, with ρ solution of (3.9)–(3.10), complemented with the initial data $\rho(0,x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} F_\epsilon(0,x,v)\,\mathrm{d}v$ (in the sense of the weak convergence in $L^1(\mathbb{R}^d)$). The convergence holds strongly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, while $\rho_\epsilon = \int_{\mathbb{R}^d} F_\epsilon\,\mathrm{d}v$ converges to ρ strongly in $L^1((0,T)\times\mathbb{R}^d)$ and in $C([0,T];L^1(\mathbb{R}^d)$-weak).

Remark 3.3.2 *Since it can be shown that the problem (3.9)–(3.10), admits a unique solution ρ for a given initial data $\rho_0 \in L^1(\mathbb{R}^d)$, the entire sequence F_ϵ converges to ρM if, in addition to (3.11), we have $\int_{\mathbb{R}^d} F_\epsilon(0,x,v)\,\mathrm{d}v \rightharpoonup \rho_0$ weakly in $L^1(\mathbb{R}^d)$.*

Stationary solutions of (3.9) satisfy

$$\nabla_x \rho_{\rm eq} + \rho_{\rm eq} \nabla_x (V + \Phi_{\rm eq}) = 0, \qquad \Phi_{\rm eq} = -\Lambda \Sigma * \rho_{\rm eq}.$$

Of course, due to the scaling the parameter c has disappeared but this problem has exactly the same form as the one discussed in Section 3.2. As a matter of fact Theorem 3.2.1 can be rephrased by saying that there exists a unique stationary solution satisfying $\int \rho_{\rm eq}\, \mathrm{d}x = \mathfrak{m}$ provided $\Lambda \|\Sigma\|_{L^\infty(\mathbb{R}^d)} \mathfrak{m}$ is small enough. It can be interpreted as a condition on the coefficients σ_1, σ_2 for instance. It is therefore a natural question to wonder whether the solutions of (3.9) with a given mass converge to the corresponding stationary state.

Corollary 3.3.3 *Let $n \geq 3$. We suppose that V is uniformly convex: there exists $\alpha > 0$ such that for any $x \in \mathbb{R}^d$, and any $\xi \in \mathbb{R}^d$, we have $\sum_{i,j=1}^d \partial^2_{x_i x_j} V(x) \xi_j \xi_i \geq \alpha |\xi|^2$. We can find $\lambda_0, \kappa > 0$ such that if $\Lambda \|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)} \mathfrak{m} < \lambda_0$, any solution ρ of (3.9) with initial data $\rho_0 \geq 0$ such that*

$$\int_{\mathbb{R}^d} \rho_0 \, \mathrm{d}x = \mathfrak{m}, \qquad \int_{\mathbb{R}^d} \frac{|\rho_0 - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, \mathrm{d}x < \infty$$

satisfies

$$\int_{\mathbb{R}^d} \frac{|\rho(t,x) - \rho_{\rm eq}(x)|^2}{\rho_{\rm eq}(x)} \, \mathrm{d}x \leq e^{-\kappa t} \int_{\mathbb{R}^d} \frac{|\rho_0 - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, \mathrm{d}x.$$

The convexity assumption on V clearly implies (**A1**). Furthermore, in the case where the diffusion coefficient in (3.9) is a mere constant this condition implies the Sobolev inequality

$$C \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \frac{e^{-V(y)}}{\bar{\mu}} \, \mathrm{d}y \right|^2 \frac{e^{-V(x)}}{\bar{\mu}} \, \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla_x g|^2 \, \frac{e^{-V}}{\bar{\mu}} \, \mathrm{d}x \tag{3.12}$$

with $\bar{\mu} = \int_{\mathbb{R}^d} e^{-V} \, \mathrm{d}x$. This is indeed the simplest case where the diffusion operator

$$\nabla_x \cdot (\nabla_x \rho + \rho \nabla_x V) = \nabla_x \cdot \left(e^{-V} \nabla_x \left(\frac{\rho}{e^{-V}} \right) \right)$$

satisfies the so–called Bakry–Emery condition. Consequently, the solutions to the linear equation $\partial_t \rho = \nabla_x \cdot (\nabla_x \rho + \rho \nabla_x V)$ can be shown to converge exponentially fast to the equilibrium $\mathfrak{m}\ e^{-V}/\bar{\mu}$. We refer the reader to [6] for an overview of these techniques. The remarkable fact is that the smallness condition on the parameters of the non linear problem ensures that the latter inherits the dissipative structure of the linear equation.

Proof of Theorem 3.3.1

In what follows, the initial data $(x, v) \mapsto F_\epsilon(0, x, v)$ is denoted as $F_{\epsilon,0}$. We start by establishing uniform estimates by first recalling mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d \times \mathbb{R}^d} F_\epsilon \, \mathrm{d}v \, \mathrm{d}x = 0.$$

Additionally, we have identified a entropy-energy functional which is dissipated by the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon\Big(\ln(F_\epsilon)+\frac{v^2}{2}+(V+\Phi_\epsilon)\Big)\,\mathrm{d}v\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\Big(\epsilon^2|\partial_t\Psi_\epsilon|^2+|\nabla_z\Psi_\epsilon|^2\Big)\,\mathrm{d}z\,\mathrm{d}x\right\}$$
$$=-\frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\right|^2\mathrm{d}v\,\mathrm{d}x.$$

The contributions of the terms containing $F_\epsilon\ln(F_\epsilon)$ and $F_\epsilon\Phi_\epsilon$ are not signed, we fix this now. We observe that the self-consistent potential energy can be dominated, using Sobolev's inequality [62, Th. 8.3], as follows

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\Phi_\epsilon F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\Big| &\leq \|F_\epsilon(t,\cdot)\|_{L^1}\|\Phi_\epsilon(t,\cdot)\|_{L^\infty}\\
&\leq \|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^n}\sigma_2(z)\Psi_\epsilon(t,x,z)\,\mathrm{d}z\Big|^2\mathrm{d}x\Big)^{1/2}\\
&\leq \|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^n}|\Psi_\epsilon(t,x,z)|^{2n/(n-2)}\,\mathrm{d}z\Big)^{(n-2)/n}\mathrm{d}x\Big)^{1/2}\\
&\leq \|F_{\epsilon,0}\|_{L^1}\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\nabla_z\Psi_\epsilon(t,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}\\
&\leq \|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)}+\frac{1}{4}\|\nabla_z\Psi_\epsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}.
\end{aligned}$$

Also, for a given nonnegative map $(x,v)\mapsto\omega(x,v)$ the particles entropy may be estimated as

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon|\ln(F_\epsilon)|\,\mathrm{d}v\,\mathrm{d}x &= \int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad -2\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\Big(\mathbf{1}_{\{e^{-\omega}\leq F_\epsilon\leq 1\}}+\mathbf{1}_{\{F_\epsilon<e^{-\omega}\}}\Big)\,\mathrm{d}v\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad +2\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\,\omega\,\mathrm{d}v\,\mathrm{d}x+\frac{4}{e}\int_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\omega/2}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

In particular, for $\omega(x,v)=\nu(V(x)+v^2/2)$, with $0<\nu<1/2$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon|\ln(F_\epsilon)|\,\mathrm{d}v\,\mathrm{d}x &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln(F_\epsilon)\,\mathrm{d}v\,\mathrm{d}x+\frac{4}{e}\int_{\mathbb{R}^d\times\mathbb{R}^d}e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x\\
&\quad +2\nu\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(V(x)+\frac{v^2}{2}\Big)F_\epsilon\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Gathering the previous estimates we arrive at

$$
\begin{aligned}
0 \le & \int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon |\ln(F_\epsilon)| \,\mathrm{d}v\,\mathrm{d}x + (1-2\nu)\int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(V(x)+\frac{v^2}{2}\Big) F_\epsilon \,\mathrm{d}v\,\mathrm{d}x \\
& +\frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t\Psi_\epsilon|^2 \,\mathrm{d}z\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_z\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x \\
& +\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} \Big|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
\le & \int_{\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon \ln(F_\epsilon) \,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^d} \Big(V(x)+\frac{v^2}{2}+\Phi_\epsilon\Big) F_\epsilon \,\mathrm{d}v\,\mathrm{d}x \\
& +\frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\partial_t\Psi_\epsilon|^2 \,\mathrm{d}z\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_z\Psi_\epsilon|^2\,\mathrm{d}z\,\mathrm{d}x \\
& +\frac{1}{\epsilon^2}\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} \Big|2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
& +\frac{4}{e}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x + \|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)} \\
\le & \mathscr{K}_0 + \frac{4}{e}\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\nu V(x)/2-\nu v^2/4}\,\mathrm{d}v\,\mathrm{d}x + \|F_{\epsilon,0}\|^2_{L^1}\|\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^{2n/(n+2)}(\mathbb{R}^n)}.
\end{aligned}
$$

These manipulations prove the following statement.

Proposition 3.3.4 *Let the assumptions of Theorem 3.3.1 be fulfilled. Then, the following assertions hold uniformly with respect to* $\epsilon > 0$

i) $F_\epsilon\Big(|\ln(F_\epsilon)| + V(x) + v^2\Big)$ *is bounded in* $L^\infty([0,\infty]; L^1(\mathbb{R}^d\times\mathbb{R}^d))$,

ii) $\epsilon\,\partial_t\Psi_\epsilon$ *and* $\nabla_z\Psi_\epsilon$ *are bounded in* $L^\infty([0,\infty]; L^2(\mathbb{R}^d\times\mathbb{R}^n))$,

iii) Φ_ϵ *and* $\nabla_x\Phi_\epsilon$ *are bounded in* $L^\infty((0,\infty)\times\mathbb{R}^d)$,

iv) $D_\epsilon \overset{\text{def}}{=} \frac{1}{\epsilon}(2\nabla_v\sqrt{F_\epsilon}+v\sqrt{F_\epsilon})$ *is bounded in* $L^2((0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$.

Let $0 < T < \infty$. By virtue of the Dunford–Pettis theorem, see [47, Section 7.3.2], it follows that

$$F_\epsilon \rightharpoonup F \text{ weakly in } L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)\ .$$

Furthermore, the control of the particles kinetic energy allows us to additionally justify that

$$\rho_\epsilon = \int_{\mathbb{R}^d} F_\epsilon\,\mathrm{d}v \rightharpoonup \rho = \int_{\mathbb{R}^d} F\,\mathrm{d}v \text{ weakly in } L^1((0,T)\times\mathbb{R}^d).$$

Let us integrate the kinetic equation in (3.8) with respect to the velocity variable. We get, on the one hand

$$\partial_t\rho_\epsilon + \nabla_x\cdot J_\epsilon = 0\,, \tag{3.13}$$

and, on the other hand, after multiplying the same equation by v

$$\epsilon^2\partial_t J_\epsilon + \nabla_x\cdot\mathbb{P}_\epsilon + \rho_\epsilon\nabla_x(V+\Phi_\epsilon) = -J_\epsilon. \tag{3.14}$$

In these equations we have denoted the momentum and the kinetic pressure as

$$J_\epsilon = \frac{1}{\epsilon}\int_{\mathbb{R}^d} vF_\epsilon \,\mathrm{d}v, \qquad \mathbb{P}_\epsilon = \int_{\mathbb{R}^d} v\otimes v\, F_\epsilon \,\mathrm{d}v.$$

Lemma 3.3.5 *The sequence* $\left(J_\epsilon\right)_{\epsilon>0}$ *is bounded in* $L^2(0,T;L^1(\mathbb{R}^d))$*. Furthermore, one can write* $\mathbb{P}_\epsilon = \rho_\epsilon\mathbb{I} + \mathbb{R}_\epsilon$*, where* $\mathbb{R}_\epsilon \to 0$ *as* $\epsilon \to 0$ *strongly in* $L^2(0,T;L^1(\mathbb{R}^d))$ *.*

Proof. Note that the momentum can be written as

$$J_\epsilon = \int_{\mathbb{R}^d} D_\epsilon \sqrt{F_\epsilon}\,\mathrm{d}v.$$

Thus, it can be estimated by a direct application of the Cauchy-Schwarz inequality and Proposition 3.3.4. Similarly, for the kinetic pressure we write

$$\begin{aligned}
\mathbb{P}_\epsilon &= \int_{\mathbb{R}^d} v\sqrt{F_\epsilon}\otimes(v\sqrt{F_\epsilon} + 2\nabla_v\sqrt{F_\epsilon})\,\mathrm{d}v - \int_{\mathbb{R}^d} v\otimes\nabla_v F_\epsilon\,\mathrm{d}v \\
&= \underbrace{\epsilon\int_{\mathbb{R}^d} v\sqrt{F_\epsilon}\otimes D_\epsilon\,\mathrm{d}v}_{\stackrel{\text{def}}{=}\mathbb{R}_\epsilon} + \mathbb{I}\int_{\mathbb{R}^d} F_\epsilon\,\mathrm{d}v
\end{aligned}$$

where we have used integration by parts in the last integral. The Cauchy-Schwarz inequality allows us to estimate

$$\int_0^T \left(\int_{\mathbb{R}^d} |\mathbb{R}_\epsilon|\,\mathrm{d}x\right)^2 \mathrm{d}t \le \epsilon\left(\sup_t \int_{\mathbb{R}^d\times\mathbb{R}^d} v^2 F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right)\left(\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d} |D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right),$$

from which one concludes using the bounds of Proposition 3.3.4. ■

Owing to Lemma 3.3.5, we can assume assume that J_ϵ admits a limit J, say in $\mathscr{M}^1((0,T)\times\mathbb{R}^d)$. We also note that $\rho_\epsilon\nabla_x\Phi_\epsilon$ is bounded in $L^1((0,T)\times\mathbb{R}^d)$ by virtue of Proposition 3.3.4. Thus, letting ϵ decrease towards 0 in (3.13) and (3.14) yields

$$\begin{aligned}
&\partial_t\rho + \nabla_x\cdot J = 0, \\
&-(J + \nabla_x\rho + \rho\nabla_x V) = \lim_{\epsilon\to 0}\rho_\epsilon\nabla_x\Phi_\epsilon.
\end{aligned}$$

Thus, it only remains the task of identifying the limit of the nonlinear term in the last equation. By using the estimates in Proposition 3.3.4 and Sobolev's embedding theorem, we can also assume that Ψ_ϵ admits a weak limit, say in $L^2((0,T)\times\mathbb{R}^d;L^{2n/(n-2)}(\mathbb{R}^n))$. In the limit $\epsilon\to 0$ the wave equation in (3.8) becomes

$$-\Delta_z\Psi(t,x,z) = -\sigma_2(z)\int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y.$$

Therefore, it follows that $\Psi(t,x,z) = -\Upsilon(z)\ \int_{\mathbb{R}^d}\sigma_1(x-y)\rho(t,y)\,\mathrm{d}y$, and as a consequence, the self-consistent potential converges to

$$\Phi(t,x) = -\Lambda\int_{\mathbb{R}^d}\Sigma(x-y)\rho(t,y)\,\mathrm{d}y\,,$$

say, weakly-$\star$ in $L^\infty((0,T)\times\mathbb{R}^d)$. A similar conclusion applies to $\nabla_x\Phi(t,x)$. Furthermore, owing to the regularity of σ_1 ($\sigma_1\in W^{2,\infty}(\mathbb{R}^d)$), we have the following property

$$\sup_{\epsilon>0}|\nabla_x\Phi_\epsilon(t,x+h)-\nabla_x\Phi_\epsilon(t,x)|\xrightarrow[|h|\to 0]{}0.$$

Using equation (3.13) and Lemma 3.3.5 it follows that $\partial_t\rho_\epsilon$ is bounded in $L^1([0,T];W^{-1,1}(\mathbb{R}^d))$. Combining these properties it is possible to apply directly the compactness statement given in [63, Lemma 5.1, p. 12] which ensures that $\rho_\epsilon\nabla_x\Phi_\epsilon\rightharpoonup\rho\nabla_x\Phi$ weakly in $L^1((0,T)\times\mathbb{R}^d)$. Thus, we conclude that

$$-(J+\nabla_x\rho+\rho\nabla_x V)=\rho\nabla_x\Phi,\qquad \Phi=-\Lambda\int_{\mathbb{R}^d}\Sigma(x-y)\rho(t,y)\,\mathrm{d}y\,.$$

Note also that the bound on $\partial_t\rho_\epsilon$ also implies that ρ_ϵ is compact in $C([0,T];L^1(\mathbb{R}^d)\text{-weak})$, that is to say the family $\left\{\int_{\mathbb{R}^d}\rho_\epsilon(t,x)\chi(x)\,\mathrm{d}x,\ \epsilon>0\right\}$ is relatively compact in $C([0,T])$ for any fixed $\chi\in L^\infty(\mathbb{R}^d)$. In particular, the initial data also makes sense for the limiting equation. With these arguments, we have justified the convergence of ρ_ϵ to ρ, solution of (3.9)–(3.10).

In fact, by using techniques elaborated in [36, 69] it is possible to improve the nature of the convergence and to show that ρ_ϵ converges strongly to ρ in $L^1((0,T)\times\mathbb{R}^d)$ and F_ϵ converges strongly to ρM in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. The reasoning combines renormalization and average lemma techniques. One of the main difficulties relies on the fact that a suitable version of the average lemma is not available for a sequence of particle distribution functions weakly compact in L^1 solving a kinetic equation with velocity derivatives in the right hand side. Let us sketch the arguments. We start by setting $\beta_\delta(s)=\frac{s}{1+\delta s}$. On the one hand, by virtue of the equi-integrability of $\left(F_\epsilon\right)_{\epsilon>0}$ we have

$$\lim_{\delta\to 0}\sup_{\epsilon>0}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}|\beta_\delta(F_\epsilon)-F_\epsilon|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t=0. \tag{3.15}$$

Indeed, observe that

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}|\beta_\delta(F_\epsilon)-F_\epsilon|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\delta F_\epsilon^2}{1+\delta F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{F_\epsilon\le\mu}\frac{\delta F_\epsilon^2}{1+\delta F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+\int_{F_\epsilon\ge\mu}F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\le \frac{\delta\mu}{1+\delta\mu}\sup_{\epsilon>0}\int F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t+\sup_{\epsilon>0}\int_{F_\epsilon\ge\mu}F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The last term can be made arbitrarily small choosing μ sufficiently large, and then, we let δ decrease towards zero. On the other hand, we shall use the renormalized equation

$$(\epsilon\partial_t+v\cdot\nabla_x)\beta_\delta(F_\epsilon)=h_{\delta,\epsilon}+\nabla_v\cdot g_{\delta,\epsilon}$$

where

$$\begin{aligned} g_{\delta,\epsilon} &= \nabla_x(V+\Phi_\epsilon)\beta_\delta(F_\epsilon) + \frac{1}{\epsilon}\beta'_\delta(F_\epsilon)M\nabla_v\left(\frac{F_\epsilon}{M}\right) \\ &= \nabla_x(V+\Phi_\epsilon)\beta_\delta(F_\epsilon) + \beta'_\delta(F_\epsilon)\sqrt{F_\epsilon}\, D_\epsilon\,, \\ h_{\delta,\epsilon} &= -\frac{1}{\epsilon}\beta''_\delta(F_\epsilon)M\nabla_v\left(\frac{F_\epsilon}{M}\right)\cdot\nabla_v F_\epsilon \\ &= -2F_\epsilon\beta''_\delta(F_\epsilon)\; D_\epsilon\cdot\nabla_v\sqrt{F_\epsilon}\,. \end{aligned}$$

In these equations we have used the fact

$$\frac{1}{\epsilon}M\nabla_v\left(\frac{F_\epsilon}{M}\right) = \frac{1}{\epsilon}(\nabla_v F_\epsilon + vF_\epsilon) = \sqrt{F_\epsilon}\, D_\epsilon.$$

For any $\delta > 0$ fixed, the sequence $\Big(\beta_\delta(F_\epsilon)\Big)_{\epsilon>0}$ is bounded in $L^1 \cap L^\infty((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Since $s \mapsto \beta'_\delta(s)$ is bounded, the sequence $\Big(g_{\delta,\epsilon}\Big)_{\epsilon>0}$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Moreover, using integration by parts one notices that

$$\begin{aligned} &\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v\sqrt{F_\epsilon}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \frac{\epsilon^2}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} v^2F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad - \int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} v\sqrt{F_\epsilon}\cdot\nabla_v\sqrt{F_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le \frac{\epsilon^2}{4}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} |D_\epsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t + \frac{d}{2}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d} F_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Therefore $\nabla_v\sqrt{F_\epsilon}$ is bounded in $L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$, and, since $s \mapsto s\beta''_\delta(s)$ is bounded, (with a bound depending on δ), the sequence $\Big(h_{\delta,\epsilon}\Big)_{\epsilon>0}$ is bounded in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. The average lemma then leads to the following compactness property

$$\sup_{\epsilon>0}\int_{(0,T)\times\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\beta_\delta(F_\epsilon)(t,x+h,v)\zeta(v)\,\mathrm{d}v - \int_{\mathbb{R}^d}\beta_\delta(F_\epsilon)(t,x,v)\zeta(v)\,\mathrm{d}v\right|\chi(t,x)\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[|h|\to 0]{} 0,$$

which holds for any $0 < T < \infty$, $\zeta \in C_c^\infty(\mathbb{R}^d)$ and nonnegative $\chi \in C_c^\infty((0,\infty)\times\mathbb{R}^d)$. We refer the reader to [32, Th. 3, Th. 6], [74, Th. 2] and [69, Appendix B]. Since the transport operator has an ϵ in front of the time derivative, the average lemma provides a gain with respect to the space variable only. Using the fact that $(V(x)+v^2)F_\epsilon$ is bounded in L^1, with $V(x) \to \infty$ as $|x| \to \infty$, we can extend previous compactness property on the whole space and for any bounded test function, non necessarily compactly supported. Hence, by using (3.15), it holds that

$$\sup_{\epsilon>0}\int_{(0,T)\times\mathbb{R}^d}\Big|\rho_\epsilon(t,x+h) - \rho_\epsilon(t,x)\Big|\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[|h|\to 0]{} 0.$$

We conclude to the strong convergence of sequence $(\rho_\epsilon)_{\epsilon>0}$ by combining this information and the fact that $(\partial_t\rho_\epsilon)_{\epsilon>0}$ is bounded in $L^1\Big([0,T];W^{-1,1}(\mathbb{R}^d)\Big)$, see Appendix 3.5. The convergence of $(F_\epsilon)_{\epsilon>0}$ towards ρM is proved by using the estimate on D_ϵ, the logarithmic Sobolev inequality and the Cziszar-Kullback inequality. Indeed, it is now clear that $\rho_\epsilon M$ tends to ρM strongly in $L^1((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)$. Therefore it remains to show that

$$\lim_{\epsilon\to 0}\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d}|F_\epsilon(t,x,v)-\rho_\epsilon(t,x)M(v)|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t=0.$$

The Csiszar-Kullback-Pinsker inequality [24], [58], implies that

$$\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|F_\epsilon-\rho_\epsilon M|\,\mathrm{d}v\,\mathrm{d}x\right)^2\leq\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right)\mathrm{d}v\,\mathrm{d}x.$$

By using the logarithmic Sobolev inequality, see e.g. [62, Theorem 8.14], the integrand of the right hand side is itself dominated as follows

$$\begin{aligned}0\leq\int_{\mathbb{R}^d}\left\{\frac{F_\epsilon}{\rho_\epsilon M}\ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right)-\frac{F_\epsilon}{\rho_\epsilon M}+1\right\}\rho_\epsilon\ M\,\mathrm{d}v\quad&=\quad\int_{\mathbb{R}^d\times\mathbb{R}^d}F_\epsilon\ln\left(\frac{F_\epsilon}{\rho_\epsilon M}\right)\mathrm{d}v\\&\leq\quad 2\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla_v\sqrt{\frac{F_\epsilon}{M}}\right|^2 M\,\mathrm{d}v\\&\leq\quad\frac{\epsilon^2}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}|D_\epsilon|^2\,\mathrm{d}v.\end{aligned}$$

Integrating with respect to space and time variable we conclude that

$$\int_{(0,T)\times\mathbb{R}^d\times\mathbb{R}^d}|F_\epsilon-\rho_\epsilon M|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\leq\frac{\epsilon\sqrt{T}}{2}\|D_\epsilon\|_{L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)}.$$

Since $\|D_\epsilon\|_{L^2((0,T)\times\mathbb{R}^d\times\mathbb{R}^d)}$ is bounded uniformly with respect to ϵ, it ends the proof. ∎

Proof of Corollary 3.3.3

We start by rewriting (3.9) as follows

$$\partial_t\rho-\nabla_x\cdot\Big(\rho_{\text{eq}}\nabla_x\Big(\frac{\rho}{\rho_{\text{eq}}}\Big)\Big)-\nabla_x\cdot\Big(\rho\nabla_x(\Phi-\Phi_{\text{eq}})\Big)=0.$$

Furthermore, we observe that

$$\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\text{eq}}|^2}{\rho_{\text{eq}}}\,\mathrm{d}x=\int_{\mathbb{R}^d}\frac{\rho^2}{\rho_{\text{eq}}}\,\mathrm{d}x-2\int_{\mathbb{R}^d}\rho\,\mathrm{d}x+\int_{\mathbb{R}^d}\rho_{\text{eq}}\,\mathrm{d}x=\int_{\mathbb{R}^d}\frac{\rho^2}{\rho_{\text{eq}}}\,\mathrm{d}x-\mathfrak{m}.$$

Therefore, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\text{eq}}|^2}{\rho_{\text{eq}}}\,\mathrm{d}x\quad&=\quad\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{\rho^2}{\rho_{\text{eq}}}\,\mathrm{d}x\\&=\quad-\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\text{eq}}}\Big)\right|^2\rho_{\text{eq}}\,\mathrm{d}x-\int_{\mathbb{R}^d}\rho\nabla_x(\Phi-\Phi_{\text{eq}})\cdot\nabla_x\Big(\frac{\rho}{\rho_{\text{eq}}}\Big)\,\mathrm{d}x.\end{aligned}$$

The last integral can be cast as

$$
\begin{aligned}
&\Lambda \int_{\mathbb{R}^d} \rho \nabla_x \Sigma * (\rho - \rho_{\rm eq}) \cdot \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \, {\rm d}x \\
&= \Lambda \int_{\mathbb{R}^d} \rho_{\rm eq} \nabla_x \Sigma * (\rho - \rho_{\rm eq}) \cdot \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \, {\rm d}x + \Lambda \int_{\mathbb{R}^d} (\rho - \rho_{\rm eq}) \nabla_x \Sigma * (\rho - \rho_{\rm eq}) \cdot \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \, {\rm d}x,
\end{aligned}
$$

where we denote by I and J the two terms of this splitting, respectively. We have, on the one hand,

$$
\begin{aligned}
|{\rm I}| &\leq \Lambda \| \nabla_x \Sigma * (\rho - \rho_{\rm eq}) \|_{L^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \rho_{\rm eq} \, {\rm d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \right)^{1/2} \\
&\leq \Lambda \| \nabla_x \Sigma \|_{L^\infty(\mathbb{R}^d)} \| \rho - \rho_{\rm eq} \|_{L^1(\mathbb{R}^d)} \sqrt{\mathfrak{m}} \left(\int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \right)^{1/2} \\
&\leq \Lambda \| \nabla_x \Sigma \|_{L^\infty(\mathbb{R}^d)} \mathfrak{m} \left(\int_{\mathbb{R}^d} \frac{|\rho - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, {\rm d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \right)^{1/2},
\end{aligned}
$$

and, on the other hand,

$$
|{\rm J}| \leq 2\Lambda \| \nabla_x \Sigma \|_{L^\infty(\mathbb{R}^d)} \mathfrak{m} \left(\int_{\mathbb{R}^d} \frac{|\rho - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, {\rm d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \right)^{1/2},
$$

since $\| \nabla_x \Sigma * (\rho - \rho_{\rm eq}) \|_{L^\infty(\mathbb{R}^d)} \| \leq \| \nabla_x \Sigma \|_{L^\infty(\mathbb{R}^d)} \| (\| \rho \|_{L^1(\mathbb{R}^d)} + \| \rho_{\rm eq} \|_{L^1(\mathbb{R}^d)})$. Thus, we arrive at the inequality

$$
\begin{aligned}
&\frac{1}{2} \frac{\rm d}{{\rm d}t} \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, {\rm d}x + \int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \\
&\qquad \leq 3\Lambda \| \nabla_x \Sigma \|_{L^\infty(\mathbb{R}^d)} \mathfrak{m} \left(\int_{\mathbb{R}^d} \frac{|\rho - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, {\rm d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x \right)^{1/2}.
\end{aligned}
$$

The final step relies on the following statement.

Lemma 3.3.6 *There exists a constant $\Omega > 0$ such that*

$$
\Omega \int_{\mathbb{R}^d} \left| g(x) - \int_{\mathbb{R}^d} g(y) \frac{\rho_{\rm eq}(y)}{\mathfrak{m}} \, {\rm d}y \right|^2 \frac{\rho_{\rm eq}(x)}{\mathfrak{m}} \, {\rm d}x \leq \int_{\mathbb{R}^d} |\nabla_x g(x)|^2 \frac{\rho_{\rm eq}(x)}{\mathfrak{m}} \, {\rm d}x
$$

holds.

Indeed, owing to Lemma 3.3.6, we get

$$
\Omega \int_{\mathbb{R}^d} \frac{|\rho - \rho_{\rm eq}|^2}{\rho_{\rm eq}} \, {\rm d}x \leq \int_{\mathbb{R}^d} \left| \nabla_x \Big(\frac{\rho}{\rho_{\rm eq}} \Big) \right|^2 \rho_{\rm eq} \, {\rm d}x.
$$

Let us denote $\mathscr{A} = 6\Lambda\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}$. By using Cauchy-Schwarz and Young inequalities, for any $0 < \nu < 2$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\\
&\quad\le(-2+\nu)\int_{\mathbb{R}^d}\left|\nabla_x\Big(\frac{\rho}{\rho_{\mathrm{eq}}}\Big)\right|^2\rho_{\mathrm{eq}}\,\mathrm{d}x+\frac{\mathscr{A}^2}{4\nu}\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\\
&\quad\le\Big((-2+\nu)\Omega+\frac{\mathscr{A}^2}{4\nu}\Big)\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x.
\end{aligned}$$

Optimizing with respect to ν yields $\nu = \frac{\mathscr{A}}{2\sqrt{\Omega}}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x\le\Big(-2+\frac{\mathscr{A}\sqrt{\Omega}}{2}\Big)\int_{\mathbb{R}^d}\frac{|\rho-\rho_{\mathrm{eq}}|^2}{\rho_{\mathrm{eq}}}\,\mathrm{d}x.$$

Therefore, we conclude ρ converges to ρ_{eq} with exponential rate $\kappa = 2\Omega - \frac{\mathscr{A}\sqrt{\Omega}}{2}$ provided the smallness condition $\mathscr{A} \le 4\sqrt{\Omega}$ is fulfilled. It finishes the proof of Corollary 3.3.3, up to the justification of Lemma 3.3.6. ■

Proof of Lemma 3.3.6. Lemma 3.3.6 is an extension of (3.12) for the state ρ_{eq}, which is seen as a perturbation of $\frac{\mathfrak{m}}{\bar{\mu}}e^{-V}$. Since $\Lambda\Sigma * \rho_{\mathrm{eq}} \ge 0$, we can write

$$\begin{aligned}
\int_{\mathbb{R}^d}|\nabla_x g|^2\rho_{\mathrm{eq}}\,\mathrm{d}x &= \frac{1}{Z_{\mathrm{eq}}}\int_{\mathbb{R}^d}|\nabla_x g|^2e^{-V+\Lambda\Sigma*\rho_{\mathrm{eq}}}\,\mathrm{d}x\\
&\ge \frac{1}{Z_{\mathrm{eq}}}\int_{\mathbb{R}^d}|\nabla_x g|^2e^{-V}\,\mathrm{d}x\\
&\ge \frac{C}{Z_{\mathrm{eq}}}\int_{\mathbb{R}^d}\left|g(x)-\int_{\mathbb{R}^d}g(y)\frac{e^{-V(y)}}{\bar{\mu}}\,\mathrm{d}y\right|^2e^{-V(x)}\,\mathrm{d}x,
\end{aligned}$$

by using (3.12). Next, $\Lambda\Sigma * \rho_{\mathrm{eq}} \le \Lambda\|\Sigma * \rho_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}^d)} \le \Lambda\|\Sigma\|_{L^\infty(\mathbb{R}^d)}\mathfrak{m}$ implies

$$\begin{aligned}
\int_{\mathbb{R}^d}|\nabla_x g|^2\rho_{\mathrm{eq}}\,\mathrm{d}x &\ge \frac{C}{Z_{\mathrm{eq}}e^{\Lambda\|\Sigma*\rho_{\mathrm{eq}}\|_{L^\infty(\mathbb{R}^d)}}}\int_{\mathbb{R}^d}\left|g(x)-\int_{\mathbb{R}^d}g(y)\frac{e^{-V(y)}}{\bar{\mu}}\,\mathrm{d}y\right|^2e^{-V(x)+\Lambda\Sigma*\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x\\
&\ge \Omega\int_{\mathbb{R}^d}\left|g(x)-\int_{\mathbb{R}^d}g(y)\frac{e^{-V(y)}}{\bar{\mu}}\,\mathrm{d}y\right|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x
\end{aligned}$$

with $\Omega = Ce^{-\Lambda\mathfrak{m}\|\Sigma\|_{L^\infty(\mathbb{R}^d)}}$. However, $\frac{\rho_{\mathrm{eq}}(x)}{\mathfrak{m}}\,\mathrm{d}x$ is a probability measure and we can check that, for any probability measure $\mathrm{d}\mu$, the function $X \mapsto \int_{\mathbb{R}^d}|g(x)-X|^2\,\mathrm{d}\mu(x)$ reaches its minimum for $X = \int_{\mathbb{R}^d}g(x)\,\mathrm{d}\mu(x)$. We conclude that

$$\int_{\mathbb{R}^d}|\nabla_x g|^2\rho_{\mathrm{eq}}\,\mathrm{d}x\ge\Omega\int_{\mathbb{R}^d}\left|g(x)-\int_{\mathbb{R}^d}g(y)\rho_{\mathrm{eq}}(y)\,\mathrm{d}y\right|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x$$

holds.

We point out the fact that Ω depends on Λ, Σ and $\mathfrak{m}$, and the condition $\mathscr{A} \le 4\sqrt{\Omega}$ met above still can be interpreted as a smallness condition for the product $\Lambda\|\Sigma\|_{W^{1,\infty}(\mathbb{R}^d)}\mathfrak{m}$, since $X \mapsto Xe^X$ tends to 0 as $X \to 0$. ■

3.4 Asymptotic trend to equilibrium

We restrict the discussion to the case where, given the total mass $\mathfrak{m}$, the equilibrium $\mathscr{M}_{\mathrm{eq}}$ is uniquely defined. We rewrite the problem by considering fluctuation

$$F = \mathscr{M}_{\mathrm{eq}} + f, \qquad \Phi = \Phi_{\mathrm{eq}} + \phi, \qquad \Psi = \Psi_{\mathrm{eq}} + \psi$$

where we remind the reader that $\rho_{\mathrm{eq}}(x) = (2\pi)^{d/2} Z[\Phi_{\mathrm{eq}}] e^{-V(x)-\Phi_{\mathrm{eq}}(x)}$, and

$$\Psi_{\mathrm{eq}}(x,z) = -\frac{1}{c^2}\Upsilon(z)\sigma_1 * \rho_{\mathrm{eq}}(x), \quad \text{with } -c^2\Delta_z\Upsilon = \sigma_2.$$

We define the operator

$$T_{\mathrm{eq}} = v\cdot\nabla_x - \nabla_x(\Phi_{\mathrm{eq}} + V)\cdot\nabla_v.$$

We obtain the coupled system for the fluctuations

$$\begin{aligned}
&(\partial_t + T_{\mathrm{eq}} - L)f = \nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}} + \nabla_x\phi\cdot\nabla_v f,\\
&(\partial^2_{tt} - c^2\Delta_z)\psi(t,x,z) = -\sigma_2(z)\int_{\mathbb{R}^d}\sigma_1(x-y)\varrho(t,y)\,\mathrm{d}y\,,\\
&\phi(t,x) = \int_{\mathbb{R}^n\times\mathbb{R}^d}\sigma_1(x-y)\sigma_2(z)\psi(t,y,z)\,\mathrm{d}z\,\mathrm{d}y
\end{aligned} \tag{3.16}$$

where $\varrho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. The problem is complemented with initial conditions

$$f\Big|_{t=0} = f_0 = F_0 - \mathscr{M}_{\mathrm{eq}}, \qquad (\psi,\partial_t\psi)\Big|_{t=0} = (\psi_0,\psi_1).$$

Note that from the definition we have

$$\|\varrho(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|f(t,\cdot)\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)} \le 2\,\mathfrak{m}, \qquad \int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

A crucial role is played by the entropy dissipation, which casts as follows

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f^2}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x \;&=\; -\gamma\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathscr{M}_{\mathrm{eq}}\Big|\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\Big|^2\,\mathrm{d}v\,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Let us now introduce the following useful notation and observations:

- For $f \in L^1(\mathbb{R}^d)$, let $\langle f\rangle \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d} f\,\mathrm{d}v$,
- and $Pf(v) \stackrel{\mathrm{def}}{=} \langle f\rangle\, M(v)$ stands for the projection onto $\mathrm{Ker}(L)$.
- Entropy dissipation makes $L^2\Big(\mathbb{R}^d\times\mathbb{R}^d;\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\Big)$ a suitable functional space, and we denote $(\cdot|\cdot)$ its inner product.
- Since we work with fluctuation we consider the closed subspace

  $$H = \Big\{f\in L^2\Big(\mathbb{R}^d\times\mathbb{R}^d;\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\Big), \int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}v\,\mathrm{d}x = (f|\mathscr{M}_{\mathrm{eq}}) = 0\Big\}$$

  endowed with the norm $\|f\|_H = \sqrt{(f|f)}$.

- We also remark that

$$T_{\rm eq}^\star = -T_{\rm eq}, \qquad P^\star = P, \qquad PT_{\rm eq}P = 0. \tag{3.17}$$

The last equality in (3.17) comes from the fact that $\langle vM\rangle = 0$ after noticing that

$$T_{\rm eq}Pf(x,v) = vM(v)\cdot\Big(\nabla_x\langle f\rangle(x) + \langle f\rangle(x)\nabla_x(\Phi_{\rm eq}+V)(x)\Big).$$

The idea of the proof of Theorem 3.2.3 consists in constructing a new functional $\mathscr{H}$ such that $\mathscr{H} \simeq \|f\|_H^2$ and identifying some number $\theta > 0$ with $\frac{\rm d}{{\rm d}t}\mathscr{H} \leq -\theta\mathscr{H}$. Such an inequality can be obtained in the linear framework, see [34]; in our case, however, the inequality will contain remainder terms that can be controlled by assuming c sufficiently large. The new functional is constructed by involving a certain operator A that combines appropriately the projection P and the transport operator $T_{\rm eq}$.

Lemma 3.4.1 ([34]) *Define the operator* $A \stackrel{\rm def}{=} (1+(T_{\rm eq}P)^\star(T_{\rm eq}P))^{-1}(T_{\rm eq}P)^\star$. *Then, we have*

- $\mathrm{Ran}(A) \subset \mathrm{Ran}(P) \subset \mathrm{Ker}(L)$, *so that* $LA = 0$ *and* $PA = A$.

- $\|Af\|_H \leq \frac{1}{2}\|(1-P)f\|_H$ *and* $\|T_{\rm eq}Af\|_H \leq \|(1-P)f\|_H$.

Proof. For the sake of completeness we collect the arguments from [34]. Owing to (3.17), we can rewrite

$$A = -(I - PT_{\rm eq}^2P)^{-1}PT_{\rm eq}.$$

Let us denote $Af = g$, thus,

$$g - PT_{\rm eq}^2Pg = -PT_{\rm eq}f \tag{3.18}$$

which can be cast as $g = P(T_{\rm eq}^2Pg - T_{\rm eq}f)$. It proves $g \in \mathrm{Ran}(P)$, and thus $LA = 0$. Furthermore, by using (3.17), we get

$$\begin{aligned}
\|g\|_H^2 + \|T_{\rm eq}Pg\|_H^2 = (g - PT_{\rm eq}^2Pg|g) &= -(PT_{\rm eq}f|g) \\
&= (f|T_{\rm eq}Pg) = ((1-P)f|T_{\rm eq}Pg) + (Pf|T_{\rm eq}Pg) \\
&= ((1-P)f|T_{\rm eq}Pg) + 0 \leq \|(1-P)f\|_H\|T_{\rm eq}Pg\|_H \\
&\leq \frac{1}{2\alpha^2}\|(1-P)f\|_H^2 + \frac{\alpha^2}{2}\|T_{\rm eq}Pg\|_H^2.
\end{aligned}$$

It yields (by successively taking $\alpha = \sqrt{2}$ and $\alpha = 1$)

$$\begin{aligned}
&\|Af\|_H = \|g\|_H \leq \frac{1}{2}\|(1-P)f\|_H, \\
&\|T_{\rm eq}Af\|_H = \|T_{\rm eq}PAf\|_H = \|T_{\rm eq}Pg\|_H \leq \|(1-P)f\|_H.
\end{aligned}$$

■

In contrast to [34], here we are dealing with a nonlinear problem. In order to handle the nonlinear terms involving fluctuations, additional estimates on the adjoint operator $A^\star$ will be needed. In what follows, we will frequently use the following simple fact: let $x \mapsto U(x)$ be a field depending only on the space variable, then

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} |vM(v)\cdot U(x)|^2 \frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)} &= \int_{\mathbb{R}^d} \underbrace{\Big(\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v\Big)}_{=\mathbb{I}} U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}\\ &= \int_{\mathbb{R}^d} \frac{|U(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x.\end{aligned}$$

Lemma 3.4.2 *The following estimates hold for the adjoint operator*

$$\|\nabla_v A^\star f\|_H \le \sqrt{\frac{d+1}{2}}\ \|f\|_H\,,\ \ \textit{and}\ \ \|vA^\star f\|_H \le \sqrt{\frac{d+2}{2}}\ \|f\|_H.$$

Proof. We have $A^\star = T_{\mathrm{eq}}P(I-PT_{\mathrm{eq}}^2P)^{-1}$. Let $g=(I-PT_{\mathrm{eq}}^2P)^{-1}f$, so that $A^\star f = T_{\mathrm{eq}}Pg$. We already know that

$$\frac{1}{2}\|g\|_H^2 + \|T_{\mathrm{eq}}Pg\|_H^2 \le \frac{1}{2}\|f\|_H^2$$

holds since taking the inner product of $(I-PT_{\mathrm{eq}}^2P)g = f$ with g yields $\|g\|_H^2 + \|T_{\mathrm{eq}}Pg\|_H^2 = (f|g) \le \frac{1}{2}(\|f\|_H^2 + \|g\|_H^2)$. Next, we compute

$$\begin{aligned}&T_{\mathrm{eq}}Pg(x,v) = T_{\mathrm{eq}}\Big(M(v)\langle g\rangle(x)\Big) = vM(v)\cdot U(x),\\ &\text{with}\ \ U(x) = \nabla_x\langle g\rangle(x) + \nabla_x(\Phi_{\mathrm{eq}}+V)\langle g\rangle(x).\end{aligned}\tag{3.19}$$

On the one hand, we observe that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|\langle g\rangle(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x &\le \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\frac{|g(x,v)|^2}{M(v)}\,\mathrm{d}v\Big)\Big(\int_{\mathbb{R}^d} M(v)\,\mathrm{d}v\Big)\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}\\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|g(x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x = \|g\|_H^2 \le \|f\|_H^2.\end{aligned}$$

On the other hand, since $\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v = \mathbb{I}$, it follows that

$$\begin{aligned}\|A^\star f\|_H^2 &= \|T_{\mathrm{eq}}Pg\|_H^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d} M^2(v)|v\cdot U(x)|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\\ &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d} v\otimes vM(v)\,\mathrm{d}v\Big)U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}\\ &= \int_{\mathbb{R}^d}\frac{|U(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x \le \frac{1}{2}\|f\|_H^2.\end{aligned}$$

Now, we turn to the velocity derivative

$$\begin{aligned}\nabla_v A^\star f(x,v) &= \nabla_v\Big(T_{\mathrm{eq}}Pg\Big)(x,v) = T_{\mathrm{eq}}\Big(\nabla_v Pg\Big)(x,v) + \nabla_x Pg(x,v)\\ &= T_{\mathrm{eq}}\Big(-vM(v)\langle g\rangle(x)\Big) + \nabla_x\Big(M(v)\langle g\rangle(x)\Big)\\ &= \Big(\mathbb{I}-v\otimes v\Big)M(v)\Big(\nabla_x\langle g\rangle(x) + \nabla_x(\Phi_{\mathrm{eq}}+V)\langle g\rangle(x)\Big)\\ &= \Big(\mathbb{I}-v\otimes v\Big)M(v)U(x).\end{aligned}$$

We are thus led to evaluate

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d}(\mathbb{I}-v\otimes v)M(v)U(x)\cdot(\mathbb{I}-v\otimes v)M(v)U(x)\frac{\mathrm{d}v\,\mathrm{d}x}{(2\pi)^{d/2}Z_{\mathrm{eq}}\mathscr{M}_{\mathrm{eq}}(x,v)}\\
&\qquad=\int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d}(\mathbb{I}-v\otimes v)^2M(v)\,\mathrm{d}v\right)}_{=(d+1)\mathbb{I}}U(x)\cdot U(x)\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}\\
&\qquad=(d+1)\int_{\mathbb{R}^d}|U(x)|^2\frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)}.
\end{aligned}$$

We conclude that

$$\|\nabla_v A^\star f\|_H^2=(d+1)\|T_{\mathrm{eq}}Pg\|_H^2\le\frac{d+1}{2}\|f\|_H^2.$$

Finally, we also note that

$$\begin{aligned}
\|vA^\star f\|_H^2 &= \|v\,T_{\mathrm{eq}}Pg\|_H^2=\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|vM(v)\big(v\cdot U(x)\big)\right|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d}\left|v\otimes v\,M(v)U(x)\right|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}(x,v)}\\
&= \int_{\mathbb{R}^d}\underbrace{\left(\int_{\mathbb{R}^d}(v\otimes v)^2M(v)\,\mathrm{d}v\right)}_{=(d+2)\mathbb{I}}U(x)\cdot U(x)\\
&= (d+2)\int_{\mathbb{R}^d}\frac{|U(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x=(d+2)\|T_{\mathrm{eq}}Pg\|_H^2\le\frac{d+2}{2}\|f\|_H^2.
\end{aligned}$$

∎

Let $0<\eta<1$ a suitable parameter to be determined in the sequel and define

$$\mathscr{H}\stackrel{\mathrm{def}}{=}\frac{1}{2}\|f\|_H^2+\eta(Af|f).$$

By using Lemma 3.4.1, we note that

$$\frac{1}{2}(1-\eta)\|f\|_H^2\le\mathscr{H}=\frac{1}{2}\|f\|_H^2+\eta(Af|f)\le\frac{1}{2}(1+\eta)\|f\|_H^2. \tag{3.20}$$

Furthermore, recall that $(Af|Lf)=(LAf|f)=0$ and $(Af|T_{\mathrm{eq}}f)=(T_{\mathrm{eq}}Af|f)$ since $T_{\mathrm{eq}}\mathscr{M}_{\mathrm{eq}}=0$. Thus, using the equation for the fluctuation (3.16) and the fact that time derivative commutes with the operator A we can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(Af|f) &= -(AT_{\mathrm{eq}}f|f)+(ALf|f)+(T_{\mathrm{eq}}Af|f)\\
&\quad+\Big(A[\nabla_x\phi\cdot\nabla_v(\mathscr{M}_{\mathrm{eq}}+f)]|f\Big)+\Big(Af|\nabla_x\phi\cdot\nabla_v(\mathscr{M}_{\mathrm{eq}}+f)\Big).
\end{aligned}$$

Thus, we are lead to estimate the four terms in the right hand side of the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} \tag{3.21}$$

with

$$
\begin{aligned}
\mathrm{I} &= (Lf|f) - \eta(AT_{\mathrm{eq}}Pf|f) = (Lf|f) - \eta(AT_{\mathrm{eq}}Pf|Pf),\\
\mathrm{II} &= -\eta(AT_{\mathrm{eq}}(1-P)f|f) + \eta(ALf|f) + \eta(T_{\mathrm{eq}}Af|f),\\
\mathrm{III} &= +\eta\Big(A[\nabla_x\phi\cdot\nabla_v f]|f\Big) + \eta\Big(Af|\nabla_x\phi\cdot\nabla_v f\Big),\\
\mathrm{IV} &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad +\eta\Big(A[\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}]|f\Big) + \eta\Big(Af|\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}\Big).
\end{aligned}
$$

In the right hand side of (3.21), the terms I and II already appear in the linear analysis of [34]. They are handled using the same arguments as those given in this reference.

The cornerstone of the proof consists in observing that $(Lf|f) - \eta(AT_{\mathrm{eq}}Pf|Pf)$ is the dissipative contribution, owing to Poincaré's inequalities. Indeed, on the one hand, there exists $\Xi > 0$ such that

$$
\Xi\int_{\mathbb{R}^d}\frac{|f(v) - \langle f\rangle M(v)|^2}{M(v)}\,\mathrm{d}v \le \int_{\mathbb{R}^d}\left|\nabla_v\Big(\frac{f(v)}{M(v)}\Big)\right|^2 M(v)\,\mathrm{d}v,
$$

see e.g. [6, Cor. 2.18]. We deduce that

$$
\Xi\|f - Pf\|_H^2 \le -(Lf|f). \tag{3.22}
$$

On the other hand, by assumption on the external potential V, there exists $\Xi' > 0$ such that the following Poincaré's inequality holds

$$
\Xi'\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|Pf(x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x \le \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|T_{\mathrm{eq}}Pf(x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x. \tag{3.23}
$$

To see this, note that the left hand side in inequality (3.23) recasts as

$$
\Xi'\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{M^2(v)|\langle f\rangle(x)|^2}{M(v)\rho_{\mathrm{eq}}(x)}\,\mathrm{d}v\,\mathrm{d}x = \Xi'\int_{\mathbb{R}^d}\frac{|\langle f\rangle(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x.
$$

For the right hand side in (3.23), we observe that

$$
\begin{aligned}
\int_{\mathbb{R}^d}\Big|T_{\mathrm{eq}}Pf(x,v)\Big|^2\frac{\mathrm{d}v}{M(v)} &= \int_{\mathbb{R}^d}\Big|(vM(v)\cdot\Big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x\Phi_{\mathrm{eq}}(x)\Big)\Big|^2\frac{\mathrm{d}v}{M(v)}\\
&= \int_{\mathbb{R}^d} v\otimes v\,M(v)\,\mathrm{d}v\,\Big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\mathrm{eq}}+V)(x)\Big)\cdot\Big(\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\mathrm{eq}}+V)(x)\Big)\\
&= \Big|\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\mathrm{eq}}+V)(x)\Big|^2.
\end{aligned}
$$

Then, the Poincaré inequality (3.23) writes as

$$
\Xi'\int_{\mathbb{R}^d}\frac{|\langle f\rangle(x)|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x \le \int_{\mathbb{R}^d}\frac{\Big|\nabla_x\langle f\rangle(x) + \langle f\rangle\nabla_x(\Phi_{\mathrm{eq}}+V)(x)\Big|^2}{\rho_{\mathrm{eq}}(x)}\,\mathrm{d}x.
$$

We set $u(x) = \langle f\rangle(x)e^{(\Phi_{\rm eq}+V)(x)/2}$, and (3.23) reduces to the more standard expression

$$\Xi' \int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x \le \int_{\mathbb{R}^d} \Big(|\nabla_x u|^2 + |u|^2 \Big(\frac{1}{4}|\nabla_x(\Phi_{\rm eq}+V)|^2 - \frac{1}{2}\Delta_x(\Phi_{\rm eq}+V) \Big) \Big) \,\mathrm{d}x$$

where we recognize a spectral property of the Schrödinger operator associated to the potential $\frac{1}{4}|\nabla_x(\Phi_{\rm eq}+V)|^2 - \frac{1}{2}\Delta_x(\Phi_{\rm eq}+V)$. The Poincaré inequality (3.23) is therefore a consequence of (**A2**), see [73]. The next step appeals to the following elementary statement.

Lemma 3.4.3 *Let S be a self-adjoint operator on a Hilbert space H. Assume there exists $\lambda > 0$ such that $(S\xi|\xi) \ge \lambda\|\xi\|^2$ holds for any $\xi \in H$. Then, we have that $1+S$ is invertible and $((1+S)^{-1}S\xi|\xi) \ge \frac{\lambda}{1+\lambda}\|\xi\|^2$.*

Proof. Clearly we have $((1+S)\xi|\xi) \ge (1+\lambda)\|\xi\|^2$. In particular $\|(1+S)\xi\| \ge (1+\lambda)\|\xi\|$ holds for any $\xi \in H$, which already proves that $(1+S)$ is injective. Next, supposing that $\lim_{n\to\infty}(1+S)x_n = y \in H$, the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus it converges to some $x \in H$. We get, for any $\xi \in H$,

$$((1+S)x_n|\xi) = (x_n|(1+S)\xi) \xrightarrow[n\to\infty]{} (y|\xi) = (x|(1+S)\xi) = ((1+S)x|\xi).$$

Hence $y = (1+S)x$ and $\mathrm{Ran}(1+S)$ is closed. Finally, let $\xi \in \overline{\mathrm{Ran}(1+S)}^{\perp} = \mathrm{Ran}(1+S)^{\perp}$: for any $x \in H$, we have $((1+S)x|\xi) = 0$. Using this relation with $x = \xi$, together with the coercivity estimate, proves that $\xi = 0$.
Using the coercivity estimate with $\xi = (1+S)^{-1}\zeta$, we proves easily that its inverse satisfies $\|(1+S)^{-1}\| \le \frac{1}{1+\lambda}$. Now, for any $\xi \in H$, we compute

$$((1+S)^{-1}S\xi|\xi) = ((1+S)^{-1}(1+S-1)\xi|\xi) = \|\xi\|^2 - ((1+S)^{-1}\xi|\xi).$$

The Cauchy–Schwarz inequality leads to

$$((1+S)^{-1}S\xi|\xi) \ge \|\xi\|^2 - \|\xi\|\|(1+S)^{-1}\xi\| \ge \Big(1 - \frac{1}{1+\lambda}\Big)\|\xi\|^2 = \frac{\lambda}{1+\lambda}\ \|\xi\|^2.$$

■

We now take $S = (T_{\rm eq}P)^\star T_{\rm eq}P$ in the Hilbert space $H = \mathrm{Ran}(P)$, using (3.19),(3.17) and setting $U(x) = \nabla_x\langle g\rangle(x) + \nabla_x(\Phi_{\rm eq}+V)\langle g\rangle(x)$, we get

$$\begin{aligned} Sg &= -PT_{eq}(v\cdot U(x)M(v)) \\ &= \int_{\mathbb{R}^d} \sum_{1\le i,j\le d} \frac{\partial U_i}{\partial x_i}\xi_i\xi_j M(v)\,\mathrm{d}v \otimes M \\ &= \Big(\int_{\mathbb{R}^d} |v_1|^2 M(v)\,\mathrm{d}v\Big)\nabla_x\cdot U \otimes M \\ &= \nabla_x\cdot(\nabla_x\langle g\rangle + \nabla_x(\Phi_{\rm eq}+V)\langle g\rangle)\otimes M \\ &= \nabla_x\cdot\Big(\rho_{\rm eq}\nabla_x\frac{\langle g\rangle}{\rho_{\rm eq}}\Big)\otimes M. \end{aligned}$$

Taking g_1 and g_2 in H, we have $(g_1,g_2) = (\langle g_1\rangle\otimes M, \langle g_2\rangle\otimes M)$ and we check

$$(Sg_1|g_2) = \int_{\mathbb{R}^d} \nabla_x\cdot\Big(\rho_{\rm eq}\nabla_x\frac{\langle g_1\rangle}{\rho_{\rm eq}}\Big)\langle g_2\rangle\frac{\mathrm{d}x}{\rho_{\rm eq}}. \tag{3.24}$$

The Fokker-Planck operator $\tilde{S} = \nabla_x \cdot \left(\rho_{\text{eq}} \nabla_x \frac{\cdot}{\rho_{\text{eq}}}\right)$ defined on the domain

$$D(\tilde{S}) = \left\{ \rho \in L^2(\mathbb{R}^d,\, \mathrm{d}x/\rho_{\text{eq}}) \quad \middle| \int_{\mathbb{R}^d} \nabla_x \Big| \frac{\langle \rho \rangle}{\rho_{\text{eq}}} \Big|^2 \rho_{\text{eq}} \,\mathrm{d}x < \infty, \quad \tilde{S}(\rho) \in L^2(\mathbb{R}^d,\, \mathrm{d}x/\rho_{\text{eq}}) \right\}$$

is clearly self-adjoint in $L^2(\mathbb{R}^d,\, \mathrm{d}x/\rho_{\text{eq}})$, so is S by (3.24). Consequently, using Lemma 3.4.3 it follows with (3.23) that

$$(AT_{\text{eq}}Pf|Pf) \geq \frac{\Xi'}{1+\Xi'}\|Pf\|_H^2. \tag{3.25}$$

We keep in mind previous observations for they will be used to estimate the term I. Proceeding as in [34], we obtain

$$\text{II} \leq \sqrt{C}\eta\|Pf\|_H\|(1-P)f\|_H \leq \frac{\Xi}{4}\|(1-P)f\|_H^2 + \frac{\eta^2}{\Xi}C\|Pf\|_H^2, \tag{3.26}$$

for a certain constant $C > 0$. Indeed, Cauchy-Schwarz inequality and Lemma 3.4.1 already prove that $\Big|(T_{\text{eq}}Af|f)\Big| \leq \|(1-P)f\|_H\|f\|_H$. Next, we remark that

$$PT_{\text{eq}}f(x,v) = M(v)\langle T_{\text{eq}}f\rangle(x) = M(v)\nabla_x \cdot \langle vf\rangle(x).$$

Therefore, using integration by parts we obtain

$$PT_{\text{eq}}Lf(x,v) = M(v)\nabla_x \cdot \langle vLf\rangle(x) = -M(v)\nabla_x \cdot \langle vf\rangle(x) = -PT_{\text{eq}}f(x,v).$$

Thus, it follows that $AL = -A$. Using again Lemma 3.4.1 we deduce that

$$\Big|(ALf|f)\Big| = \Big|(Af|f)\Big| \leq \frac{1}{2}\|(1-P)f\|_H\|f\|_H.$$

It remains to justify that there exists $C_1 > 0$ such that

$$\Big|(AT_{\text{eq}}(1-P)f|f)\Big| \leq C_1\|(1-P)f\|_H\|f\|_H,$$

which is the most delicate part of the proof of (3.26). The boundedness of $AT_{\text{eq}}(1-P)$ can be rephrased in terms of regularity analysis for the solution u of the elliptic problem

$$\rho_{\text{eq}}u - \nabla_x \cdot (\rho_{\text{eq}}\nabla_x u) = \rho, \qquad \rho_{\text{eq}} = \rho_{\text{eq}}(x).$$

We need to justify the regularization

$$\int_{\mathbb{R}^d} \left|\rho_{\text{eq}}\, D_x^2 u\right|^2 \mathrm{d}x \leq C \int_{\mathbb{R}^d} |\rho|^2 \,\mathrm{d}x.$$

That (**A3**) allows us to establish this inequality is the object of [34, Section 2, see in particular Prop. 5 and the comments with assumption (H4.1)]. The constant C in (3.26) can be estimated as $C = (3/2 + C_1)^2$.

We are left with the task of estimating the coupling terms III and IV. The following observation is crucial for the analysis; in particular, it will allow us to make the contribution of the nonlinear terms small. Owing to the linearity of the wave equation, we can write $\psi = \psi_{\text{I}} + \psi_{\text{S}}$

with ψ_{I} the solution of the free wave equation with (ψ_0, ψ_1) as initial data, namely

$$
\begin{aligned}
&(\partial^2_{tt} - c^2 \Delta_z)\psi_{\mathrm{I}} = 0,\\
&\psi_{\mathrm{I}}\Big|_{t=0} = \psi_0, \qquad \partial_t \psi_{\mathrm{I}}\Big|_{t=0} = \psi_1,
\end{aligned}
\tag{3.27}
$$

and ψ_{S} the solution of the wave equation with 0 as initial data and $-\sigma_2(z)\sigma_1 * \varrho(t,x)$ as source

$$
\begin{aligned}
&(\partial^2_{tt} - c^2 \Delta_z)\psi_{\mathrm{S}} = -\sigma_2(z)\sigma_1 * \varrho(t,x),\\
&\psi_{\mathrm{S}}\Big|_{t=0} = 0, \qquad \partial_t \psi_{\mathrm{I}}\Big|_{t=0} = 0.
\end{aligned}
\tag{3.28}
$$

Accordingly, the self-consistent potential ϕ splits into two parts

$$\phi = \phi_{\mathrm{I}} + \phi_{\mathrm{S}} \tag{3.29}$$

with

$$\phi_{\mathrm{I}}(t,x) = \int_{\mathbb{R}^d} \sigma_1(x-y) \left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_{\mathrm{I}}(t,y,z)\,\mathrm{d}z \right) \mathrm{d}y \tag{3.30}$$

and

$$\phi_{\mathrm{S}}(t,x) = \int_{\mathbb{R}^d} \sigma_1(x-y) \left(\int_{\mathbb{R}^n} \sigma_2(z)\psi_{\mathrm{S}}(t,y,z)\,\mathrm{d}z \right) \mathrm{d}y. \tag{3.31}$$

The latter can be rewritten as

$$
\begin{aligned}
&\phi_{\mathrm{S}}(t,x) = -\int_0^t \int_{\mathbb{R}^d} p(t-s)\Sigma(x-y)\varrho(s,y)\,\mathrm{d}y\,\mathrm{d}s,\\
&p(s) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\sin(sc|\xi|)}{c|\xi|}\ |\widehat{\sigma}_2(\xi)|^2\,\mathrm{d}\xi
\end{aligned}
\tag{3.32}
$$

where $\widehat{\cdot}$ stands for the Fourier transform (see e. .g. [77, Chap. I, formula (1.14)]). We can start with the following rough estimate, which appeals to assumptions (**A5**).

Lemma 3.4.4 *The potential can be estimated by using the following properties:*

i) Let ϕ_{I} be defined by (3.30) *with ψ_0 and ψ_1 of finite energy:*

$$\int_{\mathbb{R}^d \times \mathbb{R}^n} \left(|\psi_1(x,z)|^2 + c^2 |\nabla_z \psi_0(x,z)|^2 \right) \mathrm{d}z\,\mathrm{d}x = \mathscr{E}_{\mathrm{I}} < \infty.$$

Then, we have

$$
\begin{aligned}
|\phi_{\mathrm{I}}(t,x)| &\leq \frac{1}{c}\ \|\sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \sqrt{\mathscr{E}_{\mathrm{I}}},\\
|\nabla_x \phi_{\mathrm{I}}(t,x)| &\leq \frac{1}{c}\ \|\nabla_x \sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \sqrt{\mathscr{E}_{\mathrm{I}}}.
\end{aligned}
$$

ii) Let $\varrho \in L^\infty(0,\infty; L^1(\mathbb{R}^d))$ and let ϕ_{S} be defined by (3.32)*. Then, we have*

$$|\phi_{\mathrm{S}}(t,x)| \leq \frac{\Pi_0}{c^2} \|\Sigma\|_{L^\infty} \|\varrho\|_{L^\infty(0,t;L^1(\mathbb{R}^d))}, \qquad |\nabla_x \phi_{\mathrm{S}}(t,x)| \leq \frac{\Pi_0}{c^2} \|\nabla_x \Sigma\|_{L^\infty} \|\varrho\|_{L^\infty(0,t;L^1(\mathbb{R}^d))},$$

where

$$\Pi_0 = \frac{1}{(2\pi)^n}\int_0^\infty \Big| \int_{\mathbb{R}^n} \frac{\sin(t|\xi|)}{|\xi|}\ |\widehat{\sigma}_2(\xi)|^2 \,\mathrm{d}\xi \Big| \,\mathrm{d}t \in (0,\infty).$$

Proof. The solution of (3.27) satisfies the energy conservation

$$\int_{\mathbb{R}^d\times\mathbb{R}^n} \Big(|\partial_t \psi_{\mathrm{I}}(t,x,z)|^2 + c^2 |\nabla_z \psi_{\mathrm{I}}(t,x,z)|^2 \Big) \,\mathrm{d}z\,\mathrm{d}x = \mathscr{E}_{\mathrm{I}}.$$

Next, we use Hölder inequality together with the Sobolev inequality to estimate

$$\begin{aligned}
|\phi_{\mathrm{I}}(t,x)| \ &\leq \ \Big(\int_{\mathbb{R}^n} |\sigma_2(z)|^{2n/(n+2)}\Big)^{(n+2)/2n} \\
&\qquad \times \int_{\mathbb{R}^d} \sigma_1(x-y) \Big(\int_{\mathbb{R}^n} |\psi_{\mathrm{I}}(t,y,z)|^{2n/(n-2)}\Big)^{(n-2)/2n} \,\mathrm{d}y \\
&\leq \ \Big(\int_{\mathbb{R}^n} |\sigma_2(z)|^{2n/(n+2)}\Big)^{(n+2)/2n} \int_{\mathbb{R}^d} \sigma_1(x-y) \Big(\int_{\mathbb{R}^n} |\nabla_z \psi_{\mathrm{I}}(t,y,z)|^2\Big)^{1/2} \,\mathrm{d}y \\
&\leq \ \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \|\sigma_1\|_{L^2(\mathbb{R}^d)} \Big(\int_{\mathbb{R}^d\times\mathbb{R}^n} |\nabla_z \psi_{\mathrm{I}}(t,y,z)|^2 \,\mathrm{d}z\,\mathrm{d}y\Big)^{1/2} \\
&\leq \ \frac{1}{c}\ \|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \|\sigma_1\|_{L^2(\mathbb{R}^d)} \sqrt{\mathscr{E}_{\mathrm{I}}}.
\end{aligned}$$

We proceed similarly to estimate $\nabla_x \phi_{\mathrm{I}}$.
The estimate on ϕ_{S} is immediate once it is known that $t \mapsto p(t) \in L^1((0,\infty))$, with norm proportional to $1/c^2$. The claim is the object of [25, Lemma 4.4]. For the sake of completeness we sketch the proof in Appendix 3.5. ■

Let us proceed to control the nonlinear terms in the entropy estimate starting with term III.

- Owing to Lemma 3.4.1, we have $Af(x,v) = PAf(x,v) = \langle Af\rangle(x)M(v)$. Hence the product $\Big(Af|\nabla_x\phi\cdot\nabla_v f\Big)$ vanishes since it can be cast as

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\langle Af\rangle(x)M(v)\nabla_x\phi(x)\cdot\nabla_v f(x,v)}{\mathscr{M}_{\mathrm{eq}}(x,v)} \,\mathrm{d}v\,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^d} \langle Af\rangle(x)\nabla_x\phi(x)\cdot\Big(\int_{\mathbb{R}^d} \nabla_v f(x,v)\,\mathrm{d}v\Big) \frac{\mathrm{d}x}{\rho_{\mathrm{eq}}(x)} = 0.
\end{aligned}$$

- Next, duality implies that

$$\Big(A[\nabla_x\phi\cdot\nabla_v f]|f\Big) = -\Big(f\nabla_x\phi|\nabla_v A^\star f\Big) - \Big(f\nabla_x\phi|vA^\star f\Big).$$

  Thus, invoking Lemma 3.4.2, it is concluded that

$$|\Big(f\nabla_x\phi|\nabla_v A^\star f\Big)| \leq \Big(\sqrt{\frac{d+1}{2}} + \sqrt{\frac{d+2}{2}}\Big)\|f\|_H^2 \|\nabla_x\phi\|_{L^\infty}.$$

Coming back to the estimates in Lemma 3.4.4, we conclude

$$\mathrm{III} \leq \eta\Big(\frac{\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\mathrm{I}}}}{c} + \frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty}}{c^2}\Big)\Big(\sqrt{\frac{d+1}{2}} + \sqrt{\frac{d+2}{2}}\Big)\ \|f\|_H^2.$$

We continue now with the control of the term IV. Observe that $PA = A$ and $P(v\,\mathscr{M}_{\mathrm{eq}}) = 0$ imply $\Big(Af|\nabla_x\phi\cdot v\,\mathscr{M}_{\mathrm{eq}}\Big) = \Big(PAf|\nabla_x\phi\cdot v\,\mathscr{M}_{\mathrm{eq}}\Big) = 0$. Thus, the last term in IV vanishes

$$\Big(Af|\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}\Big) = -\Big(Af|\nabla_x\phi\cdot v\,\mathscr{M}_{\mathrm{eq}}\Big) = 0.$$

Additionally, we can use Lemma 3.4.2 to obtain

$$\begin{aligned}
\Big|(A[\nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}}]|f)\Big| &= \Big|(\mathscr{M}_{\mathrm{eq}}\nabla_x\phi|\nabla_v A^\star f + vA^\star f)\Big| \\
&\le \|\mathscr{M}_{\mathrm{eq}}\nabla_x\phi\|_H\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H \\
&\le \|\mathscr{M}_{\mathrm{eq}}\nabla_x\phi\|_H\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H \\
&\le \Big(\int_{\mathbb{R}^d}|\nabla_x\phi(t,x)|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Big)^{1/2}\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\|f\|_H.
\end{aligned} \tag{3.33}$$

Let us postpone for a moment the estimation of this last quantity and instead consider the integrals

$$-\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\mathrm{eq}}}\,\mathrm{d}v\,\mathrm{d}x. \tag{3.34}$$

Note that they are associated to the energy exchanges, since their sum (3.34) can be shown to be equal to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f\phi\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^n}(|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2)\,\mathrm{d}z\,\mathrm{d}x.$$

The Cauchy-Schwarz inequality permits us to evaluate

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} vf\cdot\nabla_x\phi\,\mathrm{d}v\,\mathrm{d}x\Big| &= \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\phi\cdot v\sqrt{\mathscr{M}_{\mathrm{eq}}}\;\frac{f}{\sqrt{\mathscr{M}_{\mathrm{eq}}}}\,\mathrm{d}v\,\mathrm{d}x\Big| \\
&\le \Big(\int_{\mathbb{R}^d}\Big|\nabla_x\phi(t,x)\Big|^2\Big(\int_{\mathbb{R}^d} v^2 M(v)\,\mathrm{d}v\Big)\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Big)^{1/2}\|f\|_H \qquad (3.35) \\
&\le \Big(\int_{\mathbb{R}^d}\Big|\nabla_x\phi(t,x)\Big|^2\rho_{\mathrm{eq}}(x)\,\mathrm{d}x\Big)^{1/2}\|f\|_H.
\end{aligned}$$

The second contribution in (3.34) can be estimated using the entropy dissipation. Indeed, note that

$$(Lf|f) = -\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\nabla_v\Big(\frac{f}{\mathscr{M}_{\mathrm{eq}}}\Big)\Big|^2\mathscr{M}_{\mathrm{eq}}\,\mathrm{d}v\,\mathrm{d}x = -\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big|\nabla_v f + vf\Big|^2\frac{\mathrm{d}v\,\mathrm{d}x}{\mathscr{M}_{\mathrm{eq}}}.$$

Therefore, using the Cauchy-Schwarz inequality and Lemma 3.4.4 we are led to

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\phi f\cdot\nabla_v\frac{f}{\mathscr{M}_{\rm eq}}\,\mathrm{d}v\,\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\phi\frac{f}{\sqrt{\mathscr{M}_{\rm eq}}}\cdot\frac{\nabla_v f+vf}{\sqrt{\mathscr{M}_{\rm eq}}}\,\mathrm{d}v\,\mathrm{d}x\Big|\\
&\le \|\nabla_x\phi\|_{L^\infty}\|f\|_H\sqrt{-(Lf|f)}\\
&\le \Big(\frac{\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\rm I}}}{c}+\frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty}}{c^2}\Big)\|f\|_H\sqrt{-(Lf|f)}\\
&\qquad\qquad \le -\frac12(Lf|f)+\frac12\Big(\frac{\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\sqrt{\mathscr{E}_{\rm I}}}{c}+\frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty}}{c^2}\Big)^2\|f\|_H^2.
\end{aligned}$$

In (3.33) and (3.35), we need to estimate $\int\mathscr{M}_{\rm eq}|\nabla_x\phi|^2\,\mathrm{d}v\,\mathrm{d}x$. Lemma 3.4.4 tells us that this quantity is uniformly bounded, but we need a more refined estimate that takes into account the finite speed of wave propagation. To this end, from now on, we restrict to the specific case $n=3$.

Lemma 3.4.5 *We assume $n=3$ and* $\operatorname{supp}(\sigma_2)\subset B(0,R_2)$.

*i) We suppose that (**A7**) is fulfilled. Let $\phi_{\rm I}$ be defined by (3.32). Then, there exists $\Gamma, S_0>0$, that depends on the assumptions on (ψ_0,ψ_1) (but that do not depend on $c\ge c_0$) such that*

$$|\nabla_x\phi_{\rm I}(t,x)|\le\Gamma\ \mathbf{1}_{\{ct\le S_0\}}(t).$$

ii) Let $\varrho\in L^\infty(0,\infty;L^1(\mathbb{R}^d))$ and let $\phi_{\rm S}$ be defined by (3.32). For $0\le t\le T<\infty$, we set

$$\tau(t)=\max\{0,t-2R_2/c\}.$$

Then, we have

$$|\nabla_x\phi_{\rm S}(t,x)|\le\frac{\Lambda}{c^2}\|\nabla_x\Sigma\|_{L^\infty}\|\varrho\|_{L^\infty(\tau(t),t;L^1(\mathbb{R}^d))}.$$

Proof. We use Kirchhoff's formula, see e. g. [38, Eq. (22), Chapter 2.4, p. 73], for the solution of (3.27)

$$\psi_{\rm I}(t,x,z)=\frac{1}{4\pi c^2t^2}\int_{|z-z'|=ct}\Big(t\,\psi_1(x,z')+\psi_0(x,z')+\nabla_z\psi_0(x,z')\cdot(z'-z)\Big)\,\mathrm{d}S(z')$$

with $\mathrm{d}S$ the Lebesgue measure on the sphere. We use the support assumption (**A7**) as follows. Observe that $\psi_{\rm I}(t,x,z)$, as a function of $z\in\mathbb{R}^3$, is supported in the annulus

$$\Big\{z\in\mathbb{R}^3,\ ct-R_{\rm I}\le|z|\le ct+R_{\rm I}\Big\}.$$

Accordingly, the product $\psi_{\rm I}(t,x,z)\sigma_2(z)$ vanishes for

$$ct\ge R_{\rm I}+R_2\stackrel{\rm def}{=}S_0\in(0,\infty).$$

Then, by using the Cauchy–Schwarz inequality, we get

$$\Big|\nabla_x\phi_{\rm I}(t,x)\Big|\le\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^3)}\bigg(\int_{\mathbb{R}^d}\int_{\mathbb{R}^3}\Big|\psi_{\rm I}(t,y,z)\Big|^2\,\mathrm{d}z\,\mathrm{d}y\bigg)^{1/2}\mathbf{1}_{\{ct\le S_0\}}.$$

Next, for estimating the L^2-norm of ψ_{I}, we use the Cauchy–Schwarz inequality again

$$\begin{aligned}|\psi_{\mathrm{I}}(t,x,z)|^2 &\le \left(\frac{1}{4\pi c^2 t^2}\right)^2 \int_{|z'|=ct} \mathrm{d}S(z') \\ &\qquad \times \int_{|z'|=ct} \Big(t\,\psi_1(x,z-z') + \psi_0(x,z-z') + \nabla_z\psi_0(x,z-z')\cdot z'\Big)^2 \,\mathrm{d}S(z') \\ &\le \frac{3(1+(1+c)t)^2}{4\pi^2 c^2 t^2} \int_{|z'|=ct} \Big(|\psi_1|^2 + |\psi_0|^2 + |\nabla_z\psi_0|^2\Big)(x,z-z')\,\mathrm{d}S(z').\end{aligned}$$

We integrate over x, z and we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^3} |\psi_{\mathrm{I}}(t,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &\qquad \le 3(1+(1+c)t)^2 \int_{|z'|=ct} \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^3} \Big(|\psi_1|^2 + |\psi_0|^2 + |\nabla_z\psi_0|^2\Big)(x,z-z')\,\mathrm{d}z\,\mathrm{d}x\right) \frac{\mathrm{d}S(z')}{4\pi c^2 t^2} \\ &\qquad \le 3(1+(1+c)t)^2 \Big(\|\psi_1\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^3)} + \|\psi_0\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^3)} + \|\nabla_z\psi_0\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^3)}\Big).\end{aligned}$$

Furthermore, since $z \mapsto \psi_0(x,z)$ is compactly supported in the ball $B(0,R_{\mathrm{I}})$, we can apply the Poincaré estimate $\|\psi_0(x,\cdot)\|^2_{L^2(\mathbb{R}^3)} \le C(R_{\mathrm{I}})\|\nabla_z\psi_0(x,\cdot)\|^2_{L^2(\mathbb{R}^3)}$ and finally we conclude that

$$\begin{aligned}\Big|\nabla_x\phi_{\mathrm{I}}(t,x)\Big| &\le \sqrt{3(1+C(R_{\mathrm{I}}))\mathscr{E}_0}\;(1+(1+c)t)\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^3)}\;\mathbf{1}_{\{ct\le S_0\}} \\ &\le \underbrace{\sqrt{3(1+C(R_{\mathrm{I}}))\mathscr{E}_0}\;\Big(1+S_0\Big(\frac{1}{c_0}+1\Big)\Big)\,\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^3)}}_{\stackrel{\mathrm{def}}{=}\Gamma}\;\mathbf{1}_{\{ct\le S_0\}}\end{aligned}$$

holds since $c \ge c_0$ (we remind the reader that $\mathscr{E}_0$ is the bound on the initial data Ψ_0, Ψ_1 supposed in (**A5**)) .

Similarly, the solution of (3.28) is given by (see [38, Eq. (44), Chapter 2.4, p. 82])

$$\psi_{\mathrm{S}}(t,x,z) = \frac{1}{4\pi c^2}\int_{|z-z'|\le ct} \frac{\sigma_2(z')\,\sigma_1 * \varrho(t-|z-z'|/c, x)}{|z-z'|}\,\mathrm{d}z'.$$

The product $\sigma_2(z)\sigma_2(z')$ does not vanish as long as $\max\{|z|,|z'|\} \le R_2$, which implies $|z - z'| \le 2R_2$. As a consequence, when the product $\sigma_2(z)\psi_{\mathrm{S}}(t,x,z)$ does not vanish only the values of the density $\varrho(s,\cdot)$ for $\tau(t) \le s \le t$ are relevant. More precisely, coming back to (3.31), we get

$$\begin{aligned}\Big|\nabla_x\phi_{\mathrm{S}}(t,x)\Big| &= \left|\frac{1}{c^2}\int_{|z-z'|\le ct} \frac{\sigma_2(z)\,\sigma_2(z')}{4\pi|z-z'|}\,\nabla_x\Sigma * \varrho(t-|z-z'|/c, x)\,\mathrm{d}z'\,\mathrm{d}z\right| \\ &\le \frac{\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}}{c^2}\sup_{\tau(t)\le s\le t}\|\varrho(s,\cdot)\|_{L^1(\mathbb{R}^d)}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\sigma_2(z)\sigma_2(z')}{4\pi|z-z'|}\,\mathrm{d}z'\,\mathrm{d}z \\ &\le \frac{\Lambda}{c^2}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\sup_{\tau(t)\le s\le t}\|\varrho(s,\cdot)\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

where we used for the last integral that $(-\Delta_z)\Upsilon = \sigma_2$, so that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\sigma_2(z)\sigma_2(z')}{4\pi|z-z'|}\,\mathrm{d}z'\,\mathrm{d}z = \int_{\mathbb{R}^3}\sigma_2\Upsilon\,\mathrm{d}z = \int_{\mathbb{R}^3}(-\Delta_z)\Upsilon\Upsilon\,\mathrm{d}z = \int_{\mathbb{R}^3}|\nabla_z\Upsilon|^2\,\mathrm{d}z = \Lambda.$$

■

Finally, with Lemma 3.4.5, we are able to estimate (3.33) and (3.35). Indeed, we shall use the obvious inequality

$$\int_{\mathbb{R}^d} |\varrho| \, \mathrm{d}x \le \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f|}{\sqrt{\mathscr{M}_{\mathrm{eq}}}} \sqrt{\mathscr{M}_{\mathrm{eq}}} \, \mathrm{d}v \, \mathrm{d}x \le \sqrt{\mathfrak{m}} \|f\|_H.$$

We arrive at (mind the condition $c > c_0$)

$$|\mathrm{IV}| \le -\frac{1}{2}(Lf|f) + \frac{Q}{c^2} \sup_{\tau(t)\le s\le t} \|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H \mathbf{1}_{\{ct\le S_0\}}$$

where we have set

$$\begin{aligned} Q &\stackrel{\text{def}}{=} \Big(1+\eta\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\Big)\Lambda\mathfrak{m}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)} \\ &\quad +\frac{1}{2}\Big(\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{6/5}(\mathbb{R}^3)}\sqrt{\mathscr{E}_{\mathrm{I}}}+\frac{2\mathfrak{m}\Pi_0\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}}{c_0}\Big)^2, \end{aligned}$$

and

$$\bar{\Gamma} = \Gamma\Big(1+\eta\sqrt{\mathfrak{m}}\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big)\Big).$$

Gathering the information all together it is concluded that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} &\le \frac{1}{2}(Lf|f) - \eta\,(AT_{\mathrm{eq}}Pf|Pf) + \frac{\Xi}{4}\|(1-P)f\|_H^2 + \\ &\qquad + \frac{\eta^2}{\Xi}C\|Pf\|^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H\mathbf{1}_{\{ct\le S_0\}} \end{aligned}$$

holds with

$$\bar{Q} = \frac{1}{c_0}Q + \eta\Big(\|\nabla_x\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{6/5}(\mathbb{R}^3)}\sqrt{\mathscr{E}_{\mathrm{I}}} + \frac{2\mathfrak{m}\Pi_0}{c_0}\|\nabla_x\Sigma\|_{L^\infty(\mathbb{R}^d)}\Big)\Big(\sqrt{\frac{d+1}{2}}+\sqrt{\frac{d+2}{2}}\Big).$$

Poincaré inequalities, see (3.22) and (3.25), allow us to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H} &\le -\frac{\Xi}{2}\|(1-P)f\|_H^2 - \eta\frac{\Xi'}{1+\Xi'}\Big(1-\eta\frac{C(1+\Xi')}{\Xi\Xi'}\Big)\|Pf\|_H^2 \\ &\qquad + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H\mathbf{1}_{\{ct\le S_0\}}. \end{aligned}$$

Choosing η sufficiently small ($0<\eta<\min\left\{1,\frac{\Xi\Xi'}{C(1+\Xi')}\right\}$), we can use (3.20) to define $\theta = \theta(\eta) > 0$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathscr{H}(t) &\le -2\theta\|f(t,\cdot)\|_H^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \bar{\Gamma}\|f\|_H\mathbf{1}_{\{ct\le S_0\}} \\ &\le -2\theta\|f(t,\cdot)\|_H^2 + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \theta\|f\|_H^2 + \frac{\bar{\Gamma}^2}{4\theta}\mathbf{1}_{\{ct\le S_0\}} \\ &\le -\frac{2\theta}{1-\eta}\mathscr{H}(t) + \frac{\bar{Q}}{c}\sup_{\tau(t)\le s\le t}\|f(s,\cdot)\|_H^2 + \frac{\bar{\Gamma}^2}{4\theta}\mathbf{1}_{\{ct\le S_0\}}. \end{aligned}$$

This last inequality is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\bar\theta t}\mathscr{H}(t)\Big) \le \frac{\bar Q}{c} e^{\bar\theta t} \sup_{\tau(t)\le s\le t} \|f(s,\cdot)\|_H^2 + \frac{\bar\Gamma^2}{4\theta} e^{\bar\theta t}\mathbf{1}_{\{ct\le S_0\}},$$

where we have set $\bar\theta = \frac{2\theta}{1-\eta}$. We integrate over $0 \le t \le \tau$ and we make use of (3.20) again to obtain

$$\frac{1-\eta}{2}\, e^{\bar\theta\tau}\|f(\tau,\cdot)\|_H^2 \le e^{\bar\theta\tau}\mathscr{H}(\tau) \le \mathscr{H}(0) + \frac{\bar\Gamma^2}{4\theta\bar\theta}(e^{\bar\theta S_0/c}-1) + \frac{\bar Q}{c}\int_0^\tau e^{\bar\theta s} \sup_{\tau(s)\le\sigma\le s}\|f(\sigma,\cdot)\|_H^2 \,\mathrm{d}s.$$

Setting

$$M(t) \stackrel{\text{def}}{=} \sup_{0\le s\le t} e^{\bar\theta s}\|f(s,\cdot)\|_H^2,$$

we are led to

$$\frac{1-\eta}{2}\, M(t) \le \mathscr{H}(0) + \frac{\bar\Gamma^2}{4\theta\bar\theta}(e^{\bar\theta S_0/c}-1) + \frac{\bar Q}{c}\int_0^t e^{2\bar\theta R_2/c} M(s)\,\mathrm{d}s.$$

Grönwall lemma readily implies that the estimate

$$\|f(t,\cdot)\|_H^2 \le \frac{2}{1-\eta}\Big(\mathscr{H}(0) + \frac{\bar\Gamma^2}{4\theta\bar\theta}(e^{\bar\theta S_0/c}-1)\Big)\exp\Big(-\Big(\bar\theta - \frac{2\bar Q e^{\bar\theta R_2/c}}{(1-\eta)c}\Big)t\Big)$$

holds. This completes the proof of Theorem 3.2.3. ■

Remark 3.4.6 *The main argument in Lemma 3.4.5 relies on the evaluation of the support of the solution of the wave equation by means of Huygens' principle. The analysis can be extended to odd space dimensions $n \ge 3$, at the price of more intricate formulae for ψ_{I} and ψ_{S}, see [38, Eq. (31), Chapter 2.4, p. 77]. Details are left to the reader. Arguments that make the case $n = 3$ particularly relevant on physical grounds are presented in [17].*

3.5 Appendix

Linearized stability for the dissipationless model

By construction $\mathscr{M}_{\mathrm{eq}}(x,v)$ is still a solution of the Vlasov–Wave equation (3.1)–(3.2) in the case where $\gamma = 0$. Let us consider the *linearized problem*

$$\begin{aligned}
&(\partial_t + T_{\mathrm{eq}})f = \nabla_x\phi\cdot\nabla_v\mathscr{M}_{\mathrm{eq}} = -v\mathscr{M}_{\mathrm{eq}}\cdot\nabla_x\phi,\\
&\phi(t,x) = \sigma_1 * \Big(\int_{\mathbb{R}^n}\sigma_2(z)\psi(t,\cdot,z)\,\mathrm{d}z\Big)(x),\\
&(\partial_{tt}^2 - c^2\Delta_z)\psi(t,x,z) = -\sigma_2(z)\int_{\mathbb{R}^d\times\mathbb{R}^d}\sigma_1(x-y)f(t,y,v)\,\mathrm{d}v\,\mathrm{d}y.
\end{aligned} \tag{3.36}$$

The linear stability can be established by adapting the reasoning in [8] for the gravitational Vlasov–Poisson system.

Theorem 3.5.1 *We suppose $n \geq 3$. There exists $c_1 \geq c_0 > 0$ such that the following assertion holds true for any $c > c_1$: for any $\epsilon > 0$, there exists $\eta > 0$ such that if the initial data for* (3.36) *satisfies*

$$\|f(0,\cdot)\|_H + \|\partial_t \psi(0,\cdot)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^d)} + c\|\nabla_z \psi(0,\cdot)\|_{L^2(\mathbb{R}^n\times\mathbb{R}^d)} \leq \eta\,,$$

then, for the solution of (3.36) *we have $\|f(t,\cdot)\|_H \leq \epsilon$.*

Proof. We check that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(t,x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^d} \phi(t,x)\ f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\ + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2\Big)(t,x,z)\,\mathrm{d}z\,\mathrm{d}x\Big\} = 0.\end{aligned}$$

By using the Sobolev embedding, see [62, Th. 8.3] we can estimate the contribution of the potential energy as follows

$$\begin{aligned}\Big|\int_{\mathbb{R}^d\times\mathbb{R}^d}\phi f\,\mathrm{d}v\,\mathrm{d}x\Big| &\leq \|f(t,\cdot)\|_{L^1}\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \|f(t,\cdot)\|_H\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)} \\ &\qquad\times\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^n}|\psi(t,x,z)|^{2n/(n-2)}\,\mathrm{d}z\Big)^{(n-2)/n}\mathrm{d}x\Big)^{1/2} \\ &\leq \|f(t,\cdot)\|_H\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}\|\nabla_z\psi(t,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^n)} \\ &\leq \frac{1}{4}\|f(t,\cdot)\|_H^2 + \Big(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c}\Big)^2 c^2\|\nabla_z\psi(t,\cdot)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^n)}.\end{aligned}$$

Coming back to the energy conservation, we are led to the inequalities

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(t,x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\psi(t,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &\qquad + \Big(\frac{1}{2} - \Big(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c}\Big)^2\Big)c^2\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi(t,x,z)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &\leq \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(t,x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(t,x)\ f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2\Big)(t,x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &\leq \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(0,x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^d\times\mathbb{R}^d}\phi(0,x)\ f(0,x,v)\,\mathrm{d}v\,\mathrm{d}x \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\psi|^2 + c^2|\nabla_z\psi|^2\Big)(0,x,z)\,\mathrm{d}z\,\mathrm{d}x \\ &\leq \frac{3}{4}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|f(0,x,v)|^2}{\mathscr{M}_{\mathrm{eq}}(x,v)}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^n}|\partial_t\psi(0,x,v)|^2\,\mathrm{d}z\,\mathrm{d}x \\ &\qquad + \Big(\frac{1}{2} + \Big(\frac{\|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}}{c}\Big)^2\Big)c^2\int_{\mathbb{R}^d\times\mathbb{R}^n}|\nabla_z\psi|^2(0,x,z)\,\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

This estimate allows us to conclude by choosing $c_1 = \sqrt{2}\|\sigma_1\||_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{L^{2n/(n+2)}(\mathbb{R}^n)}$. ∎

A compactness lemma

In Section 3.3, we made use of the following claim.

Lemma 3.5.2 *Let* $\Big(u_n\Big)_{n\in\mathbb{N}}$ *be a sequence defined on* $(0,T)\times\mathbb{R}^N$ *such that*

i) *We can find a non decreasing function* $\omega:(0,\infty)\to(0,\infty)$ *such that* $\sup_n \int_0^T\int_{\mathbb{R}^N} |u_n(t,x+h)-u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t \leq \omega(|h|) \xrightarrow[|h|\to 0]{} 0,$

ii) $\partial_t u_n = \sum_{|\alpha|\leq k} \partial_x^\alpha g_n^{(\alpha)}$, *with* $\sup_{n,\alpha} \|g_n^{(\alpha)}\|_{L^1((0,T)\times\mathbb{R}^N)} = M<\infty.$

Then, $\Big(u_n\Big)_{n\in\mathbb{N}}$ *is relatively compact in* $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$.

Proof. Let $\Big(\zeta^\delta\Big)_{\delta>0}$ be a sequence of mollifiers:

$$0\leq\zeta^\delta(x)\leq 1,\quad \int\zeta^\delta(x)\,\mathrm{d}x=1,\quad \mathrm{supp}(\zeta^\delta)\subset B(0,\delta).$$

We set $u_n^\delta(t,x)=\int\zeta^\delta(x-y)u_n(t,y)\,\mathrm{d}y=\int\zeta^\delta(y)u_n(t,x-y)\,\mathrm{d}y$. Owing to i), we get

$$\int_0^T\int_{\mathbb{R}^N}|u_n^\delta(t,x)-u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t\leq\int\zeta^\delta(y)\left(\int_0^T\int_{\mathbb{R}^N}|u_n(t,x-y)-u_n(t,x)|\,\mathrm{d}x\,\mathrm{d}t\right)\mathrm{d}y\leq\omega(\delta).$$

In other words u_n^δ converges in $L^1((0,T)\times\mathbb{R}^N)$ as $\delta\to 0$, uniformly with respect to n. We are going to conclude by showing the compactness in $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^N)$ of the family $\{u_n^\delta,\ n\in\mathbb{N}\}$, for $\delta>0$ fixed. It is clear that

$$\sup_n\int_0^T\int_{\mathbb{R}^N}|u_n^\delta(t,x+h)-u_n^\delta(t,x)|\,\mathrm{d}x\,\mathrm{d}t\xrightarrow[|h|\to 0]{}0$$

holds. Next, we observe that (possibly extending the functions by 0 out of $(0,T)$)

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}|u_n^\delta(t+\tau,x)-u_n^\delta(t,x)|\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}^N}\left|\int\zeta^\delta(x-y)\left(\int_t^{t+\tau}\partial_t u_n(s,y)\,\mathrm{d}s\right)\mathrm{d}y\right|\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_{\mathbb{R}^N}\left|\sum_{|\alpha|\leq k}\int\int_t^{t+\tau}(\partial^\alpha\zeta^\delta)(x-y)\ g_n^{(\alpha)}(s,y)\,\mathrm{d}s\,\mathrm{d}y\right|\mathrm{d}x\,\mathrm{d}t\\ &\leq k\|\zeta^\delta\|_{W^{k,\infty}}\int_0^T\int_t^{t+\tau}|g_n^{(\alpha)}(s,y)|\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t\leq C\tau.\end{aligned}$$

The conclusion follows by virtue of the Kolmogorov-Riesz-Fréchet criterion [47, Th. 7.56].■

Proof of Lemma 3.4.4

Let us set

$$q(t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\sin(t|\xi|)}{|\xi|}|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi.$$

The Lebesgue theorem tells us that $t \mapsto q(t)$ is continuous on $[0,\infty)$. Since σ_2 is radially symmetric, we have

$$\begin{aligned} q(t) &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n} \int_0^\infty \sin(tr) r^{n-2} |\widehat{\sigma_2}(re_1)|^2 \, \mathrm{d}r \\ &= \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n} \int_0^\infty \frac{\cos(tr)}{t} \frac{\mathrm{d}}{\mathrm{d}r} \left[r^{n-2} |\widehat{\sigma_2}(re_1)|^2 \right] \mathrm{d}r \\ &= -\frac{|\mathbb{S}^{n-1}|}{(2\pi)^n} \int_0^\infty \frac{\sin(tr)}{t^2} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \left[r^{n-2} |\widehat{\sigma_2}(re_1)|^2 \right] \mathrm{d}r. \end{aligned}$$

Therefore, q is integrable as a consequence of the following estimate

$$|q(t)| \leq \frac{K}{t^2} \quad \text{with} \quad K = \frac{|\mathbb{S}^{n-1}|}{(2\pi)^n} \int_0^\infty \left| \frac{\mathrm{d}^2}{\mathrm{d}u^2} \left[r^{n-2} |\widehat{\sigma_2}(re_1)|^2 \right] \right| \mathrm{d}r < \infty.$$

Note added to the proof.

Since the completion of this work, we learnt that a similar analysis has been performed for the Vlasov–Poisson–Fokker–Planck system by F. Hérau and L. Thomann. The result of [51] has the same flavor, namely the existence–uniqueness of a normalized equilibrium state, obtained as a solution of a nonlinear integro-differential equation (Poisson–Emden equation), and the asymptotic trend to equilibrium, with an exponential rate. The approach is also perturbative, in the sense that the results hold provided the coupling parameter in the Poisson equation is small enough.

Chapitre 4

Les limites de champ moyen

Dans cet article écrit en collaboration avec Thierry Goudon, nous montrons la validité des équations étudiées dans les deux derniers chapitres, en faisant tendre un système fini de particules vers une densité continue de particules suivant les méthodes présentées dans [43]. Les deux équations sont traitées indépendamment. Pour la première, on démontre les résultats de convergence sous des hypothèses très générales sur V en se plaçant d'un point de vue déterministe avec encore des méthodes analogues à [33]. Pour la seconde équation on adopte un point de vue probabiliste (imposé par le mouvement brownien) et on se limite au cas où le potentiel V est de gradient lipschitz. On en profite pour montrer que cette seconde équation admet bien des solutions. On montre au passage que le chaos se propage bien en tous temps. Les arguments sont en grande partie adaptés de [80, 11].

4.1 Introduction

In [17], L. Bruneau and S. De Bièvre introduced a mathematical model describing the motion of a classical particle through a homogeneous dissipative medium. The particle, which can also be subjected to the effect of an external potential V, exchanges momentum and energy with the medium, which is thought of as a vibrating field. Denoting by m the mass of the particle and by $t \mapsto q(t)$ the position of the particle at time t, the equations of motion read

$$\left\{\begin{array}{l} m\ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \nabla_x \sigma_1(q(t)-z)\ \sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial_{tt}^2 \Psi(t,x,y) - c^2\Delta_y \Psi(t,x,y) = -\sigma_2(y)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ y\in\mathbb{R}^n. \end{array}\right. \tag{4.1}$$

Here, Ψ represents the state of the environment of the particle. It creates the potential

$$\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-z)\ \sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z \tag{4.2}$$

which, in turn, influences the trajectory of the particle. The coupling is embodied in the form factor functions σ_1 and σ_2, which are both non negative, infinitely smooth and compactly supported functions. In this approach the environment can be thought of as a continuous set of membranes that vibrate with wave speed $c > 0$ in directions ($y \in \mathbb{R}^n$) perpendicular to the particles motion ($q(t) \in \mathbb{R}^d$). The model (4.1) has a Hamiltonian structure and the following energy conservation holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{m}{2}\Big|\frac{\mathrm{d}}{\mathrm{d}t}q(t)\Big|^2 + V(q(t)) + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\Psi(t,x,y)|^2 + c^2|\nabla_y\Psi(t,x,y)|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\ + \iint_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_2(y)\sigma_1(q(t)-x))\Psi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\Bigg\} = 0.\end{aligned}$$

In [17], the existence–uniqueness of solutions of (4.1) is established, together with a deep discussion on the asymptotic behavior of the system: roughly speaking, the interaction with the vibrating field acts as a friction force on the particle. We refer the reader to [1, 26, 27, 28, 78, 59] for thorough investigation of the asymptotic properties of the model, based either on analytical treatments or on numerical evidence. The question is reminiscent of the analysis of the Lorentz gas [10, 18, 42, 44, 67]; here, hard scatterers are replaced by the soft interacting potential created by the vibrating environment.

The modeling can be readily adapted in order to consider a set of N particles, all of them interacting with the vibrating medium. We are thus led to the following system of differential equations, for $j \in \{1, ..., N\}$

$$\begin{cases} m\ddot{q}_j(t) = -\nabla V(q_j(t)) - \nabla\Phi(t, q_j(t)), \\ \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\displaystyle\sum_{k=1}^{N}\sigma_1(x-q_k(t)), \end{cases} \tag{4.3}$$

where the self–consistent potential Φ is still defined by (4.2). The system is completed by the initial data

$$(q_j(0), \dot{q}_j(0)) = (q_{0,j}, p_{0,j}) \quad \text{for } j \in \{1, ..., N\}, \qquad (\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{4.4}$$

Note that the large time behavior for a system of $N > 1$ particles is likely to be much more intricate than for a single particle as analyzed in [17]. For further purposes, it is worth observing that energy conservation still holds for the N-particles system; it takes the following form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{m}{2}\sum_{j=1}^{N}\left|\frac{\mathrm{d}}{\mathrm{d}t}q_j(t)\right|^2 + \sum_{j=1}^{N}V(q_j(t)) \\
+\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\Psi(t,x,y)|^2 + c^2|\nabla_y\Psi(t,x,y)|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\
+\sum_{j=1}^{N}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_2(y)\sigma_1(q_j(t)-x))\Psi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\Bigg\} = 0.
\end{aligned}
\tag{4.5}
$$

In [25], we have revisited the model in the framework of kinetic equations. Instead of considering a particle or a set of particles described by the position–velocity pair $t \mapsto \Big(q(t), \dot{q}(t)\Big)$, we work with the particle distribution function in phase space $f(t,x,v) \geq 0$, with $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, the position and velocity variables respectively. We are thus led to the following PDE system

$$
\begin{aligned}
&\partial_t f + v\cdot\nabla_x f - \nabla_x(V+\Phi)\cdot\nabla_v f = 0, \\
&\Big(\partial_{tt}^2\Psi - c^2\Delta_y\Psi\Big)(t,x,y) = -\sigma_2(y)\int_{\mathbb{R}^d}\sigma_1(x-z)\rho(t,z)\,\mathrm{d}z, \\
&\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v,
\end{aligned}
\tag{4.6}
$$

with (4.2) defining the interaction potential again. The system (4.6) is completed by initial data

$$
f\Big|_{t=0} = f_0, \tag{4.7}
$$

and

$$
(\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{4.8}
$$

We refer to this system as the *Vlasov–Wave system*, and we warn the reader that the wave equation for Ψ holds in a direction *transverse* to the space variable (in contrast to models inspired from the Vlasov–Maxwell system, as in [15]). Finally, it can be relevant to incorporate some dissipation effects in the Vlasov equation, as in [2], namely the kinetic equation for the particle distribution function becomes

$$
\partial_t f + v\cdot\nabla_x f - \nabla_x(V+\Phi)\cdot\nabla_v f = \gamma\nabla_v\cdot(vf+\nabla_v f), \qquad \gamma > 0, \tag{4.9}
$$

which involves the Fokker–Planck operator $\nabla_v \cdot (vf + \nabla_v f)$. We refer to this model as the *Vlasov–Wave–Fokker–Planck system*. The Fokker–Planck operator induces relaxation effects: equilibrium states can be identified (which, by the way, are also stationary solutions of the Vlasov–Wave system, see [2, Appendix A]) and the asymptotic trend to equilibrium can be established.

In this paper we wish to provide a rigorous derivation of the Vlasov–Wave system (4.6) from the equations of motion of N particles, as in (4.3), when the number of particles becomes large. We also propose a similar discussion to obtain (4.9): to this end, we need to modify the deterministic model (4.3) by introducing a drag force and particles' Brownian motion, which gives a stochastic nature to the particle model. In order to deal with the asymptotic regime of a large number of particles, the self–consistent potential has to be appropriately rescaled ("weak coupling scaling"). In Section 4.2 we briefly discuss the scaling issues and we present the interpretation of the asymptotic problem. In particular, it is relevant to consider both the empirical measure associated to the particle system and the joint probability measure of the system of N particles in the $N-$body phase space [11, 43]. In Section 4.3, we set up a few technical tools that will be necessary for the asymptotic analysis. In Section 4.4 we investigate the mean field limit $N \to \infty$ and the derivation of the Vlasov–Wave system (4.6). In Section 4.5 we obtain the Vlasov–Wave–Fokker–Planck model (4.9). In both cases the analysis relies on fine estimates on the particle trajectories. For the former case, it allows us to establish directly the convergence of the empirical measure associated to the $N-$particle system to a solution of the kinetic equation, in the spirit of Dobrushin's work [33]. For the latter case, the analysis is more intricate due to the randomness induced by the Brownian motion. Thus, we study the behavior of the marginal of the $N-$particle distribution, following the arguments introduced by Sznitmann [80]. The difficulty common to both situation is the result of the fact that the definition of the self–consistent potential (4.2) does not involve a smooth convolution with respect to the space variable only, but it is also non–local with respect to time.

4.2 Mean Field Regime and Weak Coupling Scaling

The derivation of the Vlasov–Wave system (4.6) from the model (4.3) for a set of $N \gg 1$ particles requires a certain rescaling of the interaction potential. In order to clarify the motivation of the rescaling, we start by rewriting the equations in dimensionless form. To this end, let us denote by $\mathcal{U}$ the typical value of the external potential. The dimension of $\mathcal{U}$ is mass $\times \left(\frac{\text{length}}{\text{time}}\right)^2$. With L and T the length and time units respectively, we thus set

$$V(q) = \mathcal{U}\ V'(q/\mathrm{L}),$$

where V' is a dimensionless quantity. Accordingly, we also set

$$q_j(t) = \mathrm{L}q'_j(t/\mathrm{T}), \qquad p_j(t) = \frac{\mathrm{L}}{\mathrm{T}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}q'_j\right)(t/\mathrm{T}).$$

We shall use the notation $p'_j(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}q'_j(\tau)$, so that $\ddot{q}_j(t) = \frac{\mathrm{L}}{\mathrm{T}^2}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}p'_j\right)(t/\mathrm{T})$. The self–consistent potential Φ scales like $\lambda\mathcal{U}$: $\mathcal{U}$ defines the units, while the strength of the coupling is embodied into the dimensionless parameter $\lambda > 0$. To be more specific, we denote by ℓ the unit of the variable $y \in \mathbb{R}^n$ (which is not necessarily a length) and we set

$$\Psi(t, x, y) = \bar{\psi}\ \tilde{\Psi}(t/\mathrm{T}, x/\mathrm{L}, y/\ell).$$

The coupling is defined by the product

$$\sigma_1(x)\sigma_2(y) = \bar{\sigma}\ \sigma'_1(x/\mathrm{L})\sigma'_2(y/\ell)$$

where we encapsulate the unit in the single parameter $\bar\sigma > 0$. Hence, taking into account the integration over $\mathbb{R}^d \times \mathbb{R}^n$ that defines Φ, we have

$$\lambda \mathcal{U} = \bar\sigma \bar\psi L^d \ell^n.$$

We find the dimension of $\bar\psi$ by comparing the terms in the energy balance: $m\frac{\dot q_j^2}{2}$ scales like $m\frac{\mathrm{L}^2}{\mathrm{T}^2}$, $V(q_j)$ scales like $\mathcal{U}$, and the quantities that involve the vibrating field, $|\partial_t \Psi|^2 \,\mathrm{d}x\,\mathrm{d}y$ and $c^2|\nabla_y \Psi|^2 \,\mathrm{d}x\,\mathrm{d}y$, should have the same dimension. Hence, we can set

$$\kappa \mathcal{U} = \bar\psi^2 \frac{\mathrm{L}^d \ell^n}{\mathrm{T}^2},$$

with $\kappa > 0$ dimensionless.

In order to investigate the mean field regime, we assume that *the total mass is fixed:*

$$\sum_{j=1}^{N} m = Nm = \bar m \in (0, \infty)$$

does not depend on N. In other words, we have

$$m = \bar m / N.$$

Now, we compare the weights of the contributions to the energy balance (4.5):

particles' kinetic energy	$mN\frac{\mathrm{L}^2}{\mathrm{T}^2} = \bar m \frac{\mathrm{L}^2}{\mathrm{T}^2}$,
particles' potential energy	$N\mathcal{U}$,
self consistent energy	$\lambda N \mathcal{U}$,
vibrational energy	$\frac{1}{2}\kappa \mathcal{U}\Big(1 + \frac{c^2 \mathrm{T}^2}{\ell^2}\Big)$.

Imposing all terms to be of order 1 with respect to $\mathcal{E} = \bar m \frac{\mathrm{L}^2}{\mathrm{T}^2}$ leads to the following relations

$$\mathcal{U} = \frac{1}{N} \times \mathcal{E}, \qquad \lambda = 1, \qquad \kappa = N, \qquad \frac{c^2 \mathrm{T}^2}{\ell^2} = 1.$$

Having disposed of these observations, we can rewrite (4.3) in the following dimensionless form, where, for the sake of clarity, we ditch the prime symbol,

$$\begin{aligned}
&\dot q_j(t) = p_j(t),\\
&\dot p_j = -\nabla V(q_j(t)) - \nabla \Phi(t, q_j(t))\\
&\Phi(t,x) = \frac{1}{N} \int_{\mathbb{R}^d \times \mathbb{R}^n} \sigma_1(x-z)\sigma_2(y)\tilde\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}x,\\
&\partial_{tt}^2 \tilde\Psi(t,x,y) - \Delta_y \tilde\Psi(t,x,y) = -\Lambda \sigma_2(y) \sum_{k=1}^{N} \sigma_1(x - q_k(t)),
\end{aligned}$$

where the coefficient $\Lambda > 0$ is given by

$$\Lambda = N \times \frac{\bar{\sigma}}{\bar{\psi}} \mathrm{T}^2 = N \times \bar{\sigma}\bar{\psi} L^d \ell^n \times \frac{\mathrm{T}^2}{\bar{\psi}^2 \mathrm{L}^d \ell^n} = N \times \frac{\lambda \mathcal{U}}{\kappa \mathcal{U}} = 1.$$

In fact, we rescale the field (in dimensionless variables) as follows

$$\frac{1}{N}\ \tilde{\Psi}(t,x,y) = \Psi(t,x,y).$$

Beyond the notational convenience, this actually contains the assumption that the rescaled initial data (Ψ_0, Ψ_1) for the field are of order $\mathscr{O}(1)$. With this notation we arrive at the following system

$$\begin{aligned}
&\dot{q}_j(t) = p_j(t),\\
&\dot{p}_j = -\nabla V(q_j(t)) - \nabla\Phi(t, q_j(t))\\
&\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-z)\sigma_2(y)\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}x,\\
&\partial^2_{tt}\Psi(t,x,y) - \Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \frac{1}{N}\sum_{k=1}^N \sigma_1(x - q_k(t)).
\end{aligned} \tag{4.10}$$

We warn the reader not to be confused by the fact that we are actually using the same notation for both (4.3) and the rescaled problem (4.10), bearing un mind that the asymptotic limit $N \to \infty$ will be considered for (4.10).

For obtaining the model with the Fokker–Planck operator, we add to the model (4.10) a friction force, namely a force proportional to the particles velocity, and a Brownian motion. We will thus deal with the following analog to (4.10), for $j \in \{1, ..., N\}$

$$\begin{cases}
\mathrm{d}q_j = p_j\,\mathrm{d}t,\\
\mathrm{d}p_j(t) = -\nabla V(q_j(t))\,\mathrm{d}t - \ \mathrm{d}t\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q_j(t)-z)\ \sigma_2(y)\ \nabla_x\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z\\
\qquad\qquad -\gamma p_j(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B_j(t),\\
\partial^2_{tt}\Psi(t,x,y) - \Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \dfrac{1}{N}\displaystyle\sum_{k=1}^N \sigma_1(x - q_k(t)),
\end{cases} \tag{4.11}$$

with B_j a Brownian motion on $\mathbb{R}^d$.

We wish to investigate the regime $N \to \infty$ from (4.10) and (4.11), completed by the initial condition

$$(q_j(0), p_j(0)) = (q_{0,j}, p_{0,j}) \tag{4.12}$$

and

$$(\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{4.13}$$

We shall establish this way a connection with the kinetic models (4.6) and (4.9), respectively. From the rescaled systems (4.10) and (4.11), we can define two relevant quantities.

- The empirical measure of the $N-$particle system is simply defined by

$$\widehat{\mu}^N(t) = \frac{1}{N}\sum_{k=1}^{N} \delta_{(q_k(t),p_k(t))}. \tag{4.14}$$

  A direct computation shows that $f = \widehat{\mu}^N(t)$ actually satisfies (**??**), see Lemma **??**. Assuming the convergence of the initial state $\widehat{\mu}_0^N \to f_0(x,v)$ in some suitable sense, the question we address is thus nothing but a stability property of the system (4.6) for measure valued solutions.

- Considering the initial data $\Big(q_1(0), p_1(0), ..., q_N(0), p_N(0)\Big)$ as independent random variables distributed according to the same probability measure f_0 on $\mathbb{R}^d \times \mathbb{R}^d$, we can also deal with the joint probability measure μ^N, which is a probability measure on the $N-$body phase space $(\mathbb{R}^d \times \mathbb{R}^d)^N$.

For investigating the connection between the N-particle system (4.10) and the Vlasov–Wave system, it is enough to deal with the empirical measure. However, for (4.11) the trajectory of a particle is by nature a random variable, due to the Brownian motion. Thus, even if the initial data is purely deterministic, the (q_j, p_j)'s are random variables and the empirical measure $\widehat{\mu}^N$ becomes a random variable too. For this problem, the analysis is performed by dealing with the $N-$particle measure μ^N instead, or more precisely with its first marginal. Further comments and statements on these notions can be found in the surveys [11, 43].

4.3 Technical preliminaries

Main assumptions

Let us collect here the assumptions on the parameters of the model (coupling form functions, external potential), and on the initial data. Throughout the paper, we impose

$$\begin{cases} \sigma_1 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}),\ \sigma_2 \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), \\ \sigma_1(x) \geq 0,\ \sigma_2(y) \geq 0 \text{ for any } x \in \mathbb{R}^d,\ y \in \mathbb{R}^n, \\ \sigma_1, \sigma_2 \text{ are radially symmetric.} \end{cases} \tag{\textbf{H1}}$$

$$\Psi_0, \Psi_1 \in L^2(\mathbb{R}^d \times \mathbb{R}^n). \tag{\textbf{H2}}$$

$$f_0 \geq 0, \qquad f_0 \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d). \tag{\textbf{H3}}$$

$$\begin{cases} V \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^d), \\ \text{and there exists } C \geq 0 \text{ such that } V(x) \geq -C(1+|x|^2) \text{ for any } x \in \mathbb{R}^d. \end{cases} \tag{\textbf{H4}}$$

We shall need another technical assumption on the external potential, that will be detailed in Section 4.3. In [25], the existence–uniqueness of solutions to (4.6)–(4.8) is established under this set of assumptions, with (**H3**) strengthened into $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. The extension to the framework of measure–valued solutions that we present here unifies the N-particles viewpoint and the PDE viewpoint.

An overview on the Kantorowich–Rubinstein distance

The use of the dual space $\Big(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\Big)'$ appeared naturally in the analysis of [25]; this functional framework turns out to be well–adapted to establish a well–posedness theory for the model (4.6). This is strongly related to the Kantorowich–Rubinstein distance, which can be used to make the space of the probability measures a metric space. We refer the reader to [33, 82] for a detailed introduction to this notion.

Definition 4.3.1 (Kantorowich–Rubinstein distance) *Let (S, d) be a separable metric space. Let μ, ν two probability measures on S. The Kantorowich–Rubinstein distance between μ and ν is defined by*

$$W_1(\mu,\nu) = \inf_{\pi} \left\{ \int_{S^2} d(x,y)\,\mathrm{d}\pi(x,y) \right\} = \inf_{X,Y} \mathbb{E}\left[d(X,Y)\right]$$

where the infimum is taken in the first equality over measures π having μ and ν as marginals and in the second equality over all the random variables X and Y having the probability μ and ν, respectively.

The definition of W_1 is meaningful on the whole space of probability measures on S, when the distance d is bounded on $S \times S$. When $S = \mathbb{R}^d \times \mathbb{R}^d$, we can take $d(x, y) = |x - y| \wedge 1$, where $a \wedge b = \min(a, b)$. It is well known (see [33] or [82, Chapter 6]) that, when d is bounded on $S \times S$, W_1 metrizes the tight convergence in $\mathcal{M}^1(S)$ (the weak convergence of measures seen as linear forms on the space of the continuous and bounded real valued functions on S). This result will be used to prove the compactness of the sequence of the empirical measures $(\widehat{\mu}^N)_{N\in\mathbb{N}}$ in the forthcoming Sections.

Another interpretation of W_1 is given by the following Kantorowich–Rubinstein duality formula, which makes the connection with the dual of $W^{1,\infty}$ appear [82, Theorem 5.10, Chapter 5]

$$W_1(\mu,\nu) = \sup_{\|f\|_{\mathrm{Lip}}\le 1} \left| \int_S f\,\mathrm{d}(\mu-\nu) \right|, \qquad \|f\|_{\mathrm{Lip}} = \sup_{x,y\in S} \frac{|f(x)-f(y)|}{d(x,y)}.$$

In the specific case $S = \mathbb{R}^d \times \mathbb{R}^d$ and $d(x, y) = |x - y| \wedge 1$, the Kantorowich–Rubinstein formula becomes

$$\inf_{\pi} \left\{ \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} (|x-y|\wedge 1)\,\mathrm{d}\pi(x,y) \right\} = \sup_{2\|f\|_\infty, \|\nabla f\|_\infty \le 1} \left| \int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}(\mu-\nu) \right| \tag{4.15}$$

(the infimum in the left hand side is taken over measures π having μ and ν as marginals). As a matter of fact, W_1 and the $\Big(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\Big)'$ norm are equivalent

$$\frac{1}{2}\|\mu-\nu\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'} \le W_1(\mu,\nu) \le 2\|\mu-\nu\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}.$$

As mentioned above, the distance W_1 and relation (4.15) will play a crucial role in the analysis; in order to simplify the computations, from now on, we slightly modify the definition of the norm on $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ and we set

$$\|f\|_{\mathrm{Lip}} = 2\|f\|_\infty \wedge \|\nabla f\|_\infty.$$

In what follows we will deal with measures parametrized by the time variable $\mu : t \in [0,T] \longmapsto \mu_t \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$. We will say that μ lies in $C([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) - \text{tight})$, when, for any continuous and bounded function $\chi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, we have

$$\left(t \mapsto \int \chi(x,v)\,\mathrm{d}\mu_t(x,v)\right) \in C([0,T]).$$

The natural distance induced by W_1 between two measures valued functions $\mu, \nu \in C([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) - \text{tight})$ is then given by

$$\|W_1(\mu,\nu)\|_{L^\infty(0,T)} = \sup_{0\le t\le T} W_1(\mu_t,\nu_t).$$

This distance makes $C([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) - \text{tight})$ a Banach space and we shall use the shorthand notation $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. For instance, saying that a sequence $\left(\mu_N\right)_{N\in\mathbb{N}}$ converges to μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$ means

$$\lim_{N\to\infty}\left(\sup_{0\le t\le T} W_1(\mu_N(t),\mu(t))\right) = 0,$$

and, equivalently, for any $\chi \in L^\infty \cap C(\mathbb{R}^d \times \mathbb{R}^d)$

$$\lim_{N\to\infty}\int \chi(x,v)\,\mathrm{d}\mu_N(t,x,v) = \int \chi(x,v)\,\mathrm{d}\mu(t,x,v)$$

holds uniformly over $t \in [0,T]$.

Expression of the self–consistent potential

As already remarked in [25], it is convenient to rewrite the interaction potential Φ as an integral operator acting on the macroscopic density $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v$. To this end, let us set

$$t \longmapsto p(t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{\sin(c|\xi|t)}{c|\xi|}\,|\widehat{\sigma_2}(\xi)|^2\,\mathrm{d}\xi,$$

where $\widehat{\cdot}$ stands for the Fourier transform with respect to the variable $y \in \mathbb{R}^n$. We also set

$$\Phi_0(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d} \sigma_1(x-z)\left(\widehat{\Psi_0}(z,\xi)\cos(c|\xi|t) + \widehat{\Psi_1}(z,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}\right)\,\widehat{\sigma_2}(\xi)\,\mathrm{d}z\,\mathrm{d}\xi$$

which is clearly associated to the solution of the homogeneous wave equation with initial conditions (Ψ_0,Ψ_1). Finally, we define the operator $\mathcal{L}$ which associates to a measure valued function $f : (0,\infty) \to \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$ the quantity

$$\begin{aligned} \mathcal{L}(f)(t,x) &= \int_0^t p(t-s)\left(\int_{\mathbb{R}^d} \Sigma(x-z)\rho(s,z)\,\mathrm{d}z\right)\mathrm{d}s, \\ \rho(t,x) &= \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \Sigma = \sigma_1 \underset{x}{*} \sigma_1. \end{aligned} \tag{4.16}$$

Note that the regularity of the form functions σ_1, σ_2 imply that $\mathcal{L}(f)$ is a well defined smooth function, while $f(t,\cdot)$ is only measure valued. We refer the reader to [25, Section 2] for detailed proofs of the following statements. In fact [25] assumes that $f(t,\cdot)$ is an integrable function, but the results clearly apply to the measure framework as well.

Lemma 4.3.2 *Assume* (**H1**)–(**H2**). *Let f in $C_{W_1}\Big(\mathbb{R}_+; \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)\Big)$. Then, the self consistent potential Φ defined by* (4.2) *with Ψ solution of the wave equation*

$$\partial^2_{tt}\Psi - c^2\Delta_y\Psi = -\sigma_2(y)\int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma_1(x-z)f(t,z,v)\,\mathrm{d}v\,\mathrm{d}z$$

can be recast as $\Phi = \Phi_0 - \mathcal{L}(f)$.

Lemma 4.3.3 (Estimates on the self-consistent potential) *Let $0 < T < \infty$. The following properties hold:*

i) $\mathcal{L}$ belongs to the space $\mathcal{A}_T$ of the continuous operators on $C\Big([0,T]; \Big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\Big)'\Big)$ with values in $C\Big([0,T]; W^{2,\infty}(\mathbb{R}^d)\Big)$. Its norm is evaluated as follows

$$|\!|\!|\mathcal{L}|\!|\!|_{\mathcal{A}_T} \le \|\sigma_1\|^2_{W^{3,2}(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\ \frac{T^2}{2};$$

ii) $\mathcal{L}$ belongs to the space $\mathcal{B}_T$ of the continuous operators on $C\Big([0,T]; \Big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\Big)'\Big)$ with values in $C^1\Big([0,T]; L^\infty(\mathbb{R}^d)\Big)$. Its norm is evaluated as follows

$$|\!|\!|\mathcal{L}|\!|\!|_{\mathcal{B}_T} \le \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\ \left(T + \frac{T^2}{2}\right);$$

iii) Φ_0 satisfies

$$\|\Phi_0(t,\cdot)\|_{W^{2,\infty}(\mathbb{R}^d)} \le \|\sigma_1\|_{W^{2,2}(\mathbb{R}^d)}\|\sigma_2\|_{L^2(\mathbb{R}^n)}\left(\|\Psi_0\|_{L^2(\mathbb{R}^n)} + t\|\Psi_1\|_{L^2(\mathbb{R}^n)}\right),$$

for any $0 \le t \le T$, and, moreover

$$\|\Phi_0\|_{C^1([0,T];L^\infty(\mathbb{R}^d))} \le \|\sigma_1\|_{L^2(\mathbb{R}^d)}\|\sigma_2\|_{W^{1,2}(\mathbb{R}^n)}\left(2\|\Psi_0\|_{L^2(\mathbb{R}^n)} + (1+T)\|\Psi_1\|_{L^2(\mathbb{R}^n)}\right).$$

Estimates on the characteristic curves

For an external potential V that satisfies (**H4**) and for a given function Φ in $C^0([0,\infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0,\infty); L^\infty(\mathbb{R}^d))$, we can define the characteristic curves, solutions of the ODE system

$$\begin{cases} \dot{X}(t) = \xi(t), & \dot{\xi}(t) = -\nabla V(X(t)) - \nabla\Phi(t, X(t)), \\ X(\alpha) = x_0, & \xi(\alpha) = v_0. \end{cases} \tag{4.17}$$

We denote by $\varphi_\alpha^{\Phi,\beta}(x_0,v_0)$ the solution $t\mapsto(X(t),\xi(t))$ of (4.17) at time β; it can be interpreted as the position–velocity pair at time β of a particle subjected to the force field$-\nabla(V+\Phi)$, with state (x_0,v_0) at time α. The analysis relies crucially on the properties of the solutions of the differential system (4.17).

Lemma 4.3.4 *Let V satisfy* (**H4**) *and let $\Phi\in C^0([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$.*

a) There exists a function

$$R:(\mathscr{N},t,x,v)\in[0,\infty)\times[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d\longmapsto R(\mathscr{N},t,x,v)\in[0,\infty),$$

non decreasing with respect to the first two variables, such that the solution $t\mapsto(X(t),\xi(t))$ of (4.17) *satisfies the following estimate, for any $t\in\mathbb{R}$,*

$$(X(t),\xi(t))\in B_t(x_0,v_0),$$

$$B_t(x_0,v_0)=B\left(0,R\left(\|\Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))},|t|,x_0,v_0\right)\right)\subset\mathbb{R}^d\times\mathbb{R}^d.$$

b) Moreover, we have

$$|\nabla_{x,v}\varphi_0^{\Phi,t}(x,v)|\leq\exp\left(\int_0^t(1+\|\nabla^2(V+\Phi(s))\|_{L^\infty(B_s(x,v))}\,\mathrm{d}s\right).$$

c) Taking two additional potential Φ_1 and Φ_2, the following estimate holds for any $t>0$

$$\begin{aligned}&|(\varphi_0^{\Phi_1,t}-\varphi_0^{\Phi_2,t})(x_0,v_0)|\\&\quad\leq\int_0^t\|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)}\exp\left(\int_s^t\|\nabla^2(\Phi_1(\tau)+V)\|_{L^\infty(\tilde{B}_\tau(x_0,v_0))}\,\mathrm{d}\tau\right)\mathrm{d}s,\end{aligned}$$

where we have set

$$\tilde{B}_\tau(x,v)=B\left(0,R\left(\max_{i=1,2}\|\Phi_i\|_{C^1([0,\tau];L^\infty(\mathbb{R}^d))},\tau,x,v\right)\right).$$

Proof of Lemma 4.3.4. We refer the reader to [25, Section 3] for items a) and c); we prove b) here. Let (X_1,ξ_1) and (X_2,ξ_2) stand for the solutions (4.17) with initial data (x_1,v_1) and (x_2,v_2), respectively, at $\alpha=0$. We have

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}(X_1-X_2)(t)=(\xi_1-\xi_2)(t),\\[2mm]\dfrac{\mathrm{d}}{\mathrm{d}t}(\xi_1-\xi_2)(t)=-\nabla V(X_1(t))+\nabla V(X_2(t))-\nabla\Phi(t,X_1(t))+\nabla\Phi(t,X_2(t)).\end{cases}$$

Let us set

$$K_t=\bigcup_{i=1,2}B\left(0,R\left(\|\Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))},|t|,x_i,v_i\right)\right)$$

Using a), we obtain (at least in the sense of distributions)

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}|X_1-X_2|\leq|\xi_1-\xi_2|,\\[2mm]\dfrac{\mathrm{d}}{\mathrm{d}t}|\xi_1-\xi_2|\leq\|\nabla^2(V+\Phi(t))\|_{L^\infty(K_t)}|X_1(t)-X_2(t)|.\end{cases}$$

The Grönwall Lemma yields

$$|(X_1,\xi_1)-(X_2,\xi_2)| \le |(x_1,v_1)-(x_2,v_2)| \exp\left(\int_0^t \|\nabla^2(V+\Phi(t))\|_{L^\infty(K_s)}\,\mathrm{d}s\right).$$

By definition, we have $(X_i,\xi_i)=\varphi_0^{\Phi,t}(x_i,v_i)$ for $i=1,2$. Letting (x_2,v_2) converge to (x_1,v_1), we get

$$|\nabla\varphi_0^{\Phi_1,t}(x_1,v_1)| \le \exp\left(\int_0^t \left(1+\|\nabla^2(V+\Phi(s))\|_{L^\infty(B_s(x_1,v_1))}\right)\mathrm{d}s\right),$$

by using the continuity of R. ■

We can now introduce an additional technical requirement on the external potential. Given Ψ_0 and Ψ_1 satisfying (**H2**), we set

$$r(t,x,v)=R(\|\Phi_0\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}+\|\!|\mathcal{L}|\!\|_{\mathcal{B}_t},t,x,v), \tag{4.18}$$

where R is the function defined in Lemma 4.3.4-a) and the quantity $\|\Phi_0\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}$ is well defined by virtue of Lemma 4.3.3. Then, we assume that

$$\mathscr{K}_T(\mu_0)=\int_{\mathbb{R}^d\times\mathbb{R}^d}\exp\left(\int_0^T \|\nabla^2 V\|_{L^\infty(B(0,r(t,x,v)))}\,\mathrm{d}t\right)\mathrm{d}\mu_0<\infty. \tag{\textbf{H5}}$$

We refer the reader to [25] for further comments on this assumption.

Assumptions (**H1**)–(**H5**) are supposed to be fulfilled throughout the paper.

4.4 Mean field Limit for the Vlasov–Wave system

Particle viewpoint vs. kinetic viewpoint

According to Lemma 4.3.2, it is equivalent to consider a solution (f,Ψ) to the system (4.6)-(4.8) and a solution f of

$$\begin{cases} \partial_t f+v\cdot\nabla_x f=\nabla_x\left(V+\Phi_0-\mathcal{L}(f)\right)\cdot\nabla_v f,\\ f(0,x,v)=f_0(x,v), \end{cases} \tag{4.19}$$

with $\mathcal{L}$ defined by (4.16). It allows us to establish that the empirical measure $\widehat{\mu}^N$ associated to (4.10) satisfies (4.6), with $c=1$ (since (4.10) is set in dimensionless form).

Lemma 4.4.1 *The following properties are satisfied:*

i) If Ψ and $(q_j,p_j)_{j\in\{1,\dots,N\}}$ is solution of (4.10) *with* (4.12)–(4.13) *then $\widehat{\mu}^N$ is solution of* (4.19) *with initial data $f_0=\frac{1}{N}\sum_{j=1}^N\delta_{(q_{0,j},p_{0,j})}$.*

ii) Moreover, if μ is a solution of (4.19) *with initial data $f_0=\frac{1}{N}\sum_{j=1}^N\delta_{(q_{0,j},p_{0,j})}$, then for all $t\ge 0$, we can find Ψ and $(q_j,p_j)_{j\in\{1,\dots,N\}}$ solution of* (4.10) *with* (4.12)–(4.13)*, such that μ is given by* (4.14).

Proof. We split the proof into two steps.

Step 1: Proof of i)

Let Ψ and $(q_j, p_j)_{j\in\{1,\dots,N\}}$ be a solution of (4.10) with (4.12)–(4.13). We associate to this solution the empirical measure $\widehat{\mu}^N$ given by (4.14). Let $\chi \in C \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. We have

$$\left\langle \widehat{\mu}_t^N | \chi \right\rangle = \frac{1}{N}\sum_{j=1}^N \chi(q_j(t), p_j(t)).$$

As a matter of fact, we observe that, on the one hand,

$$\left\langle \widehat{\mu}_t^N | \chi \right\rangle \Big|_{t=0} = \frac{1}{N}\sum_{j=1}^N \chi(q_{0,j}, p_{0j}),$$

and, on the other hand, the self–consistent potential can be cast as

$$\begin{aligned}
\Phi(t,x) &= \int_{\mathbb{R}^d} \sigma_1(x-z)\sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z,\\
\partial_{tt}^2\Psi - c^2\Delta_y\Psi &= -\sigma_2(y)\ \left\langle \widehat{\mu}_t^N | \sigma_1(x)\otimes \mathbf{1}(v)\right\rangle.
\end{aligned}$$

Now, let $\psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ and compute the distribution bracket

$$\begin{aligned}
&\left\langle\!\!\left\langle \Big(\partial_t + v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v\Big)\widehat{\mu}_t^N | \psi \right\rangle\!\!\right\rangle\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} - \int_0^\infty \left\langle \widehat{\mu}_t^N | \Big(\partial_t + v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v\Big)\psi(t,\cdot)\right\rangle\,\mathrm{d}t\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N\int_0^\infty \Big((\partial_t\psi)(t,q_j(t),p_j(t))\\
&\qquad + p_j(t)\cdot\nabla_x\psi(t,q_j(t),p_j(t)) - \nabla(V+\Phi)(t,q_j(t))\cdot\nabla_v\psi(t,q_j(t),p_j(t))\Big)\,\mathrm{d}t\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N\int_0^\infty \Big((\partial_t\psi)(t,q_j(t),p_j(t))\\
&\qquad + \dot{q}_j(t)\cdot\nabla_x\psi(t,q_j(t),p_j(t)) + \dot{p}_j(t)\cdot\nabla_v\psi(t,q_j(t),p_j(t))\Big)\,\mathrm{d}t\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big[\psi\Big(t,q_j(t),p_j(t)\Big)\Big]\,\mathrm{d}t\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} + \frac{1}{N}\sum_{j=1}^N \psi(0,q_{0,j},p_{0,j})\\
&= -\left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} + \left\langle \widehat{\mu}_t^N | \psi(t,\cdot)\right\rangle\Big|_{t=0} = 0.
\end{aligned}$$

It follows that $\widehat{\mu}^N$ is a weak solution of (4.6).

Step 2: Proof of ii)

Let Ψ and μ be a solution of (4.6) (with $c=1$) with initial data (Ψ_0,Ψ_1) and $\mu_0 = \frac{1}{N}\sum_{j=1}^N \delta_{(q_{0,j},p_{0,j})}$, respectively. Equivalently, μ satisfies (4.19). Then, given χ_0 in $C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, and $T\geq 0$, we define $\chi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ as to be the solution of the following Liouville equation

$$\begin{cases}
\partial_t\chi + v\cdot\nabla_x\chi - \nabla_x(V+\Phi)\cdot\nabla_v\chi = 0,\\
\chi(T,x,v) = \chi_0.
\end{cases}$$

Here, the potential Φ is given by $\Phi = \Phi_0 - \mathcal{L}(\mu)$. By virtue of Lemma 4.3.3, it is a smooth function and the solution χ can be obtained by integrating along characteristics, see (4.17). Namely, for any $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, $t \mapsto \chi(t, \varphi_0^{\Phi,t}(x, v))$ does not depend on the time variable $t \in [0, T]$ and we have

$$\chi(t, x, v) = \chi_0 \circ \varphi_t^{\Phi,T}(x, v).$$

Next, we observe that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \langle \mu(t)|\chi(t)\rangle \\ &\qquad = + \langle \mu(t)|(v \cdot \nabla_x - \nabla_x(V + \Phi) \cdot \nabla_v)\chi(t)\rangle - \langle \mu(t)|(v \cdot \nabla_x - \nabla_x(V + \Phi) \cdot \nabla_v)\chi(t)\rangle \\ &\qquad = 0.\end{aligned}$$

Integrating this relation over $[0, T]$, we obtain

$$\langle \mu(T), \chi_0\rangle = \langle \mu_0, \chi(0, \cdot)\rangle = \frac{1}{N} \sum_{j=1}^{N} \chi_0(\varphi_0^{\Phi,T}(q_{0,j}, p_{0,j})).$$

Therefore, since the final time $T \geq 0$ and the trial function χ_0 are arbitrary, we conclude that $\mu(t)$ is given by

$$\mu(t) = \frac{1}{N} \sum_{j=1}^{N} \delta_{(q_j(t), p_j(t))}, \qquad \text{with } \Big(q_j(t), p_j(t)\Big) = \varphi_0^{\Phi,t}(q_{0,j}, p_{0,j}).$$

By definition of φ^Φ, and since $\Phi = \Phi_0 - \mathcal{L}(\mu)$, we check that $(q_j, p_j)_{j\in\{1,\dots,N\}}$ satisfy (4.10).
■

It is therefore equivalent to prove the existence–uniqueness of a solution of (4.6)-(4.8), and to prove the existence–uniqueness of a solution of (4.19) with the initial data $\widehat{\mu}_0^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(q_{0,j}, p_{0,j})}$. We shall adopt the PDE viewpoint, so that we can conclude by adapting the reasoning in [25].

Existence theory for the Vlasov–Wave system

This Section is devoted to the proof of the following statement, which extends to the framework of measure–valued solutions the analysis of [25]. In particular it justifies the existence of solutions for (4.10).

Theorem 4.4.2 *Assume* (**H1**)-(**H5**). *Let* $0 < T < \infty$. *Then, there exists a unique* $\mu \in C_{W_1}([0, T]; \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$ *solution of* (4.19) *on* $[0, T]$ *such that* $\mu(0) = f_0$.

It is clear that, given $(q_{0,1}, p_{0,1}, ..., q_{0,N}, p_{0,N}) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ and $(\Psi_0, \Psi_1) \in L^2(\mathbb{R}^d \times \mathbb{R}^n)$, the condition (**H5**) is fulfilled by $\widehat{\mu}_0^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(q_{0,j}, p_{0,j})}$. As a consequence of Lemma 4.3.2 and Lemma 4.4.1, we also obtain the following claim.

Corollary 4.4.3 *For all* $N \in \mathbb{N} \setminus \{0\}$, *for all* $(q_{0,j}, p_{0,j})_{j\in\{1,...N\}}$ *in* $(\mathbb{R}^d \times \mathbb{R}^d)^N$ *and for all* Ψ_0, Ψ_1 *in* $L^2(\mathbb{R}^d \times \mathbb{R}^n)$, *there exists a unique solution* $(\Psi, q_1, p_1, ...q_N, p_N)$ *of* (4.10)–(4.13).

The proof of Theorem 4.4.2 relies on a fixed point strategy. To this end, we introduce the following mapping. For a non negative finite measure μ_0, we denote by Λ (or Λ_{μ_0} if necessary) the mapping which associates to Φ in $C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))$ the unique weak solution μ of the Liouville equation

$$\partial_t\mu + v\cdot\nabla_x\mu - \nabla_x\left(V+\Phi\right)\cdot\nabla_v\mu = 0,$$

with initial data μ_0. Owing to the regularity of V and Φ, the characteristic curves $\varphi_\alpha^{\Phi,\beta}(x_0,v_0)$ solution $t\mapsto(X(t),\xi(t))$ of (4.17), are well–defined. Thus, the solution $\mu=\Lambda_{\mu_0}(\Phi)\in C([0,\infty);\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{weak-}\star)$ is defined as the pushforward of μ_0 by the flow; namely, it is given by the following duality formula: for any $\chi\in C\cap L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, we have

$$\langle\Lambda_{\mu_0}(\Phi)(t)|\chi\rangle = \left\langle\mu_0|\chi\circ\varphi_0^{\Phi,t}\right\rangle = \int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(\varphi_0^{\Phi,t}(x,v))\,\mathrm{d}\mu_0(x,v). \tag{4.20}$$

Lemma 4.4.4 *The mapping* $(\mu_0,\Phi)\mapsto\Lambda_{\mu_0}(\Phi)$ *is continuous from the functional space* $\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\times\Big(C([0,\infty);W^{2,\infty}(\mathbb{R}^d))\cap C^1([0,\infty);L^\infty(\mathbb{R}^d))\Big)$ *to* $C_{W_1}([0,\infty);\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$.

Proof. Let $0<T<\infty$. We consider two pairs $(\mu_{0,1},\Phi_1)$ and $(\mu_{0,2},\Phi_2)$ and we wish to estimate the Kantorovich-Rubinstein distance between $\Lambda_{\mu_{0,1}}(\Phi_1)(t)$ and $\Lambda_{\mu_{0,2}}(\Phi_2)(t)$ for all $0\le t\le T$. Owing to (4.15) we have

$$\begin{aligned} W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t),\Lambda_{\mu_{0,2}}(\Phi_2)(t)\right) &= \sup_{\|\chi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\left(\chi\circ\varphi_0^{\Phi_1,t}\,\mathrm{d}\mu_{0,1}-\chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}\mu_{0,2}\right)\right| \\ &\le \sup_{\|\chi\|_{\mathrm{Lip}}\le 1}\int_{\mathbb{R}^d\times\mathbb{R}^d}\left|\chi\circ\varphi_0^{\Phi_1,t}-\chi\circ\varphi_0^{\Phi_2,t}\right|\mathrm{d}\mu_{0,1} \\ &\quad + \sup_{\|\chi\|_{\mathrm{Lip}}\le 1}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}(\mu_{0,1}-\mu_{0,2})\right|. \end{aligned} \tag{4.21}$$

In order to bound those two terms, we introduce the cut off function

$$\begin{array}{ll} \theta_R(z)=\theta(z/R), & \theta\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d), \\ |\nabla\theta(z)|\le 1 \text{ for any } z\in\mathbb{R}^d\times\mathbb{R}^d, & \theta(z)=0 \text{ for } |z|\ge 2, \\ 0\le\theta(z)\le 1 \text{ for any } z\in\mathbb{R}^d\times\mathbb{R}^d, & \theta(z)=1 \text{ for } |z|\le 1. \end{array} \tag{4.22}$$

Let $\epsilon>0$. We can find $R>0$ depending on $\mu_{0,1}$ and ϵ, such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\theta_R)\,\mathrm{d}\mu_{0,1}\le\frac{\epsilon}{4} \tag{4.23}$$

For χ in $W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$, we split the first term arising in the right hand side of (4.21) into two parts

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d}|\chi\circ\varphi_0^{\Phi_1,t}-\chi\circ\varphi_0^{\Phi_2,t}|\,\mathrm{d}\mu_{0,1} &\le \int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(z)\|\nabla\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left|\varphi_0^{\Phi_1,t}-\varphi_0^{\Phi_2,t}\right|(z)\,\mathrm{d}\mu_{0,1}(z) \\ &\quad +\frac{\epsilon}{2}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}. \end{aligned}$$

Lemma 4.3.4 allows us to control $|\varphi_0^{\Phi_1,t}-\varphi_0^{\Phi_2,t}|$. We set

$$A(\Phi_1,\mu_{0,1},\epsilon) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(z)\exp\left(\int_0^T\|\nabla^2(\Phi_1(\tau)+V)\|_{L^\infty(B_\tau(z))}\,\mathrm{d}\tau\right)\mathrm{d}\mu_{0,1}(z),$$

and we get

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |\chi\circ\varphi_0^{\Phi_1,t} - \chi\circ\varphi_0^{\Phi_2,t}|\,\mathrm{d}\mu_{0,1}$$
$$\leq A(\Phi_1,\mu_{0,1},\epsilon)\|\nabla\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\int_0^t \|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)}\,\mathrm{d}s + \frac{\epsilon}{2}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}. \quad (4.24)$$

We turn to estimate the second term of the right hand side of (4.21). Owing to (4.23) and (4.15), we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\theta_R)\,\mathrm{d}\mu_{0,2} \leq \frac{\epsilon}{4} + W_1(\mu_{0,1},\mu_{0,2}).$$

It allows us to split the integral as we did above, and we obtain

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}(\mu_{0,1}-\mu_{0,2}) \leq \int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(\chi\circ\varphi_0^{\Phi_2,t})\,\mathrm{d}(\mu_{0,1}-\mu_{0,2}) + \|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left(\frac{\epsilon}{2}+W_1(\mu_{0,1},\mu_{0,2})\right) \quad (4.25)$$

By using (4.15) we get

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(\chi\circ\varphi_0^{\Phi_2,t})\,\mathrm{d}(\mu_{0,1}-\mu_{0,2})\right|$$
$$\leq \left(2\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\wedge\|\nabla(\theta_M(\chi\circ\varphi_0^{\Phi_2,t})\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\right)W_1(\mu_{0,1},\mu_{0,2}) \quad (4.26)$$
$$\leq \left(1+B(\mu_{0,1},\epsilon)e^{\int_0^t\|\nabla^2\Phi_2(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s}\right)W_1(\mu_{0,1},\mu_{0,2})\|\chi\|_{\mathrm{Lip}}$$

where we have used Lemma 4.3.4, again, to estimate $\nabla\varphi_0^{\Phi_2,t}$ and we have set

$$B(\mu_{0,1},\epsilon) = \sup_{|z|\leq 2R}\exp\left(\int_0^T(1+\|\nabla^2 V\|_{L^\infty(B_s(z))})\,\mathrm{d}s\right).$$

Coming back to (4.21), and combining the intermediate estimates (4.24), (4.25) and (4.26), we conclude that, for all $\epsilon>0$, we can find $A(\Phi_1,\mu_1,\epsilon)$ and $B(\mu_1,\epsilon)$ such that, for all $0<t<T$, we have

$$W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t),\Lambda_{\mu_{0,2}}(\Phi_2)(t)\right) \leq \left(2+B(\mu_{0,1},\epsilon)e^{\int_0^t\|\nabla^2\Phi_2(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s}\right)W_1(\mu_{0,1},\mu_{0,2}) + A(\Phi_1,\mu_{0,1},\epsilon)\int_0^t\|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)}\,\mathrm{d}s + \frac{\epsilon}{2}.$$

(Note that $A(\Phi_1,\mu_1,\epsilon)$ and $B(\mu_1,\epsilon)$ depend on $0<T<\infty$.) Letting both $W_1(\mu_{0,2},\mu_{0,1})$ and $\|(\Phi_1-\Phi_2)(s)\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))}$ go to 0, we conclude that $W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t),\Lambda_{\mu_{0,2}}(\Phi_2)(t)\right)$ tends to 0, uniformly for $t\in[0,T]$. (Note that we should assume that Φ_1 and Φ_2 remain bounded in $C([0,T];W^{2,\infty}(\mathbb{R}^d))$ and in $C^1([0,T];W^{1,\infty}(\mathbb{R}^d))$.) ■

We are now ready to justify the existence and uniqueness of solution of (4.6)–(4.8), or equivalently of (4.19).

Proof of Theorem 4.4.2. We turn to the fixed point reasoning. For μ given in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$, we set

$$\mu\longmapsto\mathcal{T}_{\mu_0}(\mu) = \Lambda_{\mu_0}(\Phi_0-\mathcal{L}(\mu)).$$

It is clear that a fixed point of $\mathcal{T}_{\mu_0}$ is a solution to (4.19). Note also that, as a consequence of Lemma 4.3.3 and Lemma 4.4.4, $\mathcal{T}_{\mu_0}(\mu)(t)\in\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$. More precisely, we know that

$\mu \mapsto \mathcal{T}_{\mu_0}(\mu)$ is continuous with values in the space $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. We shall prove that $\mathcal{T}_{\mu_0}$ admits an iteration which is a contraction on $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. Let μ_1 and μ_2 be two elements of this set. We denote $\varphi_\alpha^{\Phi_i,t}$ the flow of (4.17) with $\Phi_i = \Phi_0 - \mathcal{L}(\mu_i)$. By using (4.21), we get

$$\begin{aligned} W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) &= \sup_{\|\chi\|_{\mathrm{Lip}}\leq 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\chi \circ \varphi_0^{\Phi_1,t} - \chi \circ \varphi_0^{\Phi_2,t}\right) \mathrm{d}\mu_0 \\ &\leq \sup_{\|\chi\|_{\mathrm{Lip}}\leq 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|\nabla\chi\|_\infty \left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right|(x,v)\,\mathrm{d}\mu_0 \\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right|(x,v)\,\mathrm{d}\mu_0. \end{aligned} \tag{4.27}$$

By using Lemma 4.3.4-b), we obtain

$$\begin{aligned} &\left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right|(x,v) \\ &\quad\leq \bar{m}_T \int_0^t \|\mathcal{L}(\mu_1-\mu_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))} \\ &\qquad\qquad \times \exp\left(\int_s^t \|\nabla^2 V\|_{L^\infty B(0,R(\|\Phi_0+\mathcal{L}(\mu_i)\|_{C^1([0,u];L^\infty(\mathbb{R}^d))},u,x_0,v_0))}\,\mathrm{d}u\right)\mathrm{d}s, \end{aligned}$$

where we have also used

$$\begin{aligned} &\exp\left(\int_0^T \|\nabla^2(\Phi_0(u) - \mathcal{L}(\mu_1)(u)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}u\right) \\ &\qquad\qquad \leq \exp\left(\int_0^T \left(\|\nabla^2\Phi_0(u)\|_{L^\infty(\mathbb{R}^d)} + \|\!|\mathcal{L}|\!\|_{\mathcal{A}_u}\right)\mathrm{d}u\right) = \bar{m}_T, \end{aligned}$$

thanks to the simplification

$$\|\mu_1\|_{C\left(0,u;(W^{1,\infty})'\right)} \leq \|\mu_1\|_{C(0,u;(L^\infty)')} \leq \|\mu_0\|_{(L^\infty)'} \leq 1.$$

Going back to (4.27) yields

$$\begin{aligned} W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) \leq\ & \bar{m}_T \int_{\mathbb{R}^d\times\mathbb{R}^d}\int_0^t \|\mathcal{L}(\mu_1-\mu_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))} \\ & \times \exp\left(\int_s^t \|\nabla^2 V\|_{L^\infty(B(0,r(u,x,v)))}\,\mathrm{d}u\right)\mathrm{d}s\,\mathrm{d}\mu_0(x,v). \end{aligned}$$

Using Lemma 4.3.3 and (4.15), it recasts as

$$W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) \leq \bar{m}'_T \mathscr{K}_T \int_0^t \left(\sup_{0\leq\tau\leq s} W_1(\mu_{1,\tau},\mu_{2,\tau})\right)\mathrm{d}s \tag{4.28}$$

with

$$\bar{m}'_T = \bar{m}_T \times \sup_{0\leq s\leq T} \|\!|\mathcal{L}|\!\|_{\mathcal{A}_s}.$$

By induction, we deduce that

$$W_1\left(\mathcal{T}^\ell_{\mu_0}(\mu_1)(t), \mathcal{T}^\ell_{\mu_0}(\mu_2)(t)\right) \leq \frac{(t\bar{m}'_T\mathscr{K}_T)^\ell}{\ell!} \sup_{0\leq t\leq T} W_1(\mu_{1,t},\mu_{2,t})$$

holds for any $\ell \in \mathbb{N}$ and $0 \leq t \leq T$. Finally, we are led to

$$\sup_{0\leq t\leq T} W_1\left(\mathcal{T}^\ell_{\mu_0}(\mu_1(t)), \mathcal{T}^\ell_{\mu_0}(\mu_2(t))\right) \leq \frac{(T\bar{m}'_T\mathscr{K}_T)^\ell}{\ell!} \|W_1(\mu_1(t),\mu_2(t))\|_{L^\infty(0,T)}.$$

It shows that an iteration of $\mathcal{T}_{\mu_0}$ is a contraction in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. Therefore, there exists a unique fixed point μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. Furthermore, the solution is continuous with respect to the parameters of the system. ∎

Asymptotic analysis

We now wish to investigate the limit $N \to \infty$ in (4.10)–(4.13) and to justify that it allows us to derive (4.6)–(4.8). Since any finite measure f_0 can be obtained as the tight–limit of sums of Dirac masses, see [64, Chap. 2, Th. 6.9], by using Lemma 4.4.1, Theorem 4.4.2 and Corollary 4.4.3, we reduce the question we address to a stability issue. We suppose that Ψ_0 and Ψ_1 do not depend on N and we consider a sequence of initial data $\left(\mu_0^N\right)_{\mathbb{N}\in\mathbb{N}}$. We associate to these data the corresponding solutions μ^N of (4.6)–(4.8). We are going to distinguish two situations. Either we suppose that

$$\begin{aligned}&(\mu_0^N)_{N\in\mathbb{N}} \text{ converges tightly to a finite measure } \mu_0 \text{ and}\\ &\mathscr{K}_T = \sup_{N\in\mathbb{N}} \left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} \exp\left(\int_0^T \|\nabla^2 V\|_{L^\infty(B(0,r(t,x,v)))}\,\mathrm{d}t \right) \mathrm{d}\mu_0^N \right\} < \infty.\end{aligned} \tag{H6}$$

where r is defined by (4.18) (which does not depend upon N), or

$$\text{the sequence } (\mu_0^N)_{N\in\mathbb{N}} \text{ is tight.} \tag{H6b}$$

Clearly (**H6**) is stronger than (**H6b**), and it allows us to obtain sharper results. The analysis of the situation with (**H6b**) only relies on a compactness analysis.

Theorem 4.4.5 (Stability for the Vlasov–Wave system) *a) Assume* (**H6**). *Then there exists a measure–valued function μ solution of* (4.19) *with initial data μ_0 such that μ^N converges to μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$.*

b) Assume **(H6b)**. *Then, we can extract a subsequence $\left(\mu^{N_\ell}\right)_{\ell\in\mathbb{N}}$ such that μ^{N_ℓ} converges to a measure–valued function μ, solution of* (4.19) *with initial data $\mu_0 = \lim_{\ell\to\infty} \mu_0^{N_\ell}$ (tightly), in $C_{W_1}([0,\infty);\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$.*

Proof.
Step 1: Proof of a). We remind the reader that the solutions of (4.19) in Theorem 4.4.2 have been obtained as fixed points of the application $\mathcal{T}$; namely, we have

$$\mu^N = \mathcal{T}_{\mu_0^N}(\mu^N) = \Lambda_{\mu_0^N}(\Phi_0 - \mathcal{L}(\mu^N)), \qquad \mu = \mathcal{T}_{\mu_0}(\mu) = \Lambda_{\mu_0}(\Phi_0 - \mathcal{L}(\mu)).$$

Based on this, we write

$$W_1(\mu_t^N, \mu_t) \leq W_1\left(\mathcal{T}_{\mu_0^N}(\mu^N)(t), \mathcal{T}_{\mu_0^N}(\mu)(t)\right) + W_1\left(\mathcal{T}_{\mu_0^N}(\mu)(t), \mathcal{T}_{\mu_0}(\mu)(t)\right). \tag{4.29}$$

Bearing in mind (4.28), (**H6**) leads to the following estimate

$$W_1\left(\mathcal{T}_{\mu_0^N}(\mu^N)(t), \mathcal{T}_{\mu_0^N}(\mu)(t)\right) \leq \bar{m}_T' \mathscr{K}_T \int_0^t \left(\sup_{0\leq\tau\leq s} W_1(\mu_\tau^N, \mu_\tau) \right) \mathrm{d}s.$$

Let us set

$$\alpha^N(t) = \sup_{0\le\tau\le t} W_1(\mu^N_\tau,\mu_\tau),$$
$$\beta^N(t) = \sup_{0\le\tau\le t} W_1\left(\mathcal{T}_{\mu_0^N}(\mu)(\tau),\mathcal{T}_{\mu_0}(\mu)(\tau)\right).$$

The previous inequality implies

$$\alpha^N(t) \le \beta^N(t) + \bar m'_T \mathscr{K}_T \int_0^t \alpha^N(s)\,\mathrm{d}s. \tag{4.30}$$

It can be cast as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\bar m'_T\mathscr{K}_T t}\int_0^t \alpha^N(\tau)\,\mathrm{d}\tau\right) \le \beta^N(t)e^{-\bar m'_T\mathscr{K}_T t}.$$

Integration over $[0,s]$ yields

$$\int_0^s \alpha^N(\tau)\,\mathrm{d}\tau \le \int_0^s \beta^N(\tau)e^{\bar m'_T\mathscr{K}_T(s-\tau)}\,\mathrm{d}\tau \le \frac{\beta^N(s)}{\bar m'_T\mathscr{K}_T}\left(e^{\bar m'_T\mathscr{K}_T s}-1\right).$$

Going back to (4.30) leads to

$$\alpha^N(t) \le \beta^N(t)e^{\bar m'_T\mathscr{K}_T t}.$$

With $\Phi = \Phi_0 - \mathcal{L}(\mu)$, by definition of β^N and $\mathcal{T}$, this can be rewritten

$$\alpha^N(t) \le e^{T\bar m'_T\mathscr{K}_T} \sup_{0\le s\le T} W_1\left(\Lambda_{\mu_0^N}(\Phi),\Lambda_{\mu_0}(\Phi)\right).$$

We conclude by coming back to Lemma 4.4.4.

Step 2: Proof of b). We start by showing that the sequence $\left(\mu^N\right)_{N\in\mathbb{N}}$ is compact in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. By hypothesis, we note that

$$\bar m = \sup_{N\in\mathbb{N}} \|\mu_0^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)} < \infty.$$

Pick $\chi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. For any $0\le t\le T$, we have, on the one hand,

$$\begin{aligned}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v)\right| &\le \|\mu_t^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\le \|\mu_0^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\ &\le \bar m\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)},\end{aligned} \tag{4.31}$$

by mass conservation, and, on the other hand,

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v)\right|\\ &\quad=\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\Big(v\cdot\nabla_x\chi-\nabla_x(V+\Phi_0-\mathcal{L}(\mu^N)(t))\cdot\nabla_v\chi\Big)(x,v)\,\mathrm{d}\mu_t^N(x,v)\right|\\ &\quad\le \bar m\bigg(\|v\cdot\nabla\chi-\nabla V\cdot\nabla_v\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d))}\\ &\qquad\qquad+\Big(\|\mathcal{L}\|_{\mathcal{A}_T}\bar m+\|\Phi_0\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}\Big)\|\chi\|_{L^\infty}\bigg).\end{aligned}$$

Lemma 4.3.3 then ensures that the set

$$\left\{t\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v),\ N\in\mathbb{N}\right\}$$

is equibounded and equicontinuous; hence, by virtue of Arzela–Ascoli's theorem it is relatively compact in $C([0,T])$. Going back to (4.31), a simple approximation argument allows us to extend the conclusion to any trial function χ in $C_0(\mathbb{R}^d \times \mathbb{R}^d)$, the space of continuous functions that vanish at infinity. This space is separable; consequently, by a diagonal argument, we can extract a subsequence, still labelled by $N \in \mathbb{N}$, and find a measure valued function $\mu \in C\Big([0,\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) - \text{weak-}\star\Big)$ such that

$$\lim_{N\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}\mu_t^N(x,v) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(x,v)\,\mathrm{d}\mu_t(x,v)$$

holds uniformly on $[0,T]$, for any $\chi \in C_0(\mathbb{R}^d \times \mathbb{R}^d)$ and $0 < T < \infty$. As a matter of fact, we note that, for any $0 \le t \le T$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}\mu_t(x,v) \le \bar{m}.$$

Next, we establish the tightness of the sequence of solutions. Let $\epsilon > 0$ be fixed once for all. Owing to (**H6b**), we can find $M_\epsilon > 0$ such that for all $N \ge 0$,

$$\int_{x^2+v^2\ge M_\epsilon^2} \mathrm{d}\mu_0^N(x,v) \le \epsilon.$$

Let us set

$$A_\epsilon = \sup\{r(T,x,v),\ (x,v) \in B(0,M_\epsilon)\}$$

where we remind the reader that $r(T,x,v)$ has been defined in (4.18): $0 < A_\epsilon < \infty$ is well defined by Lemma 4.3.3. Let $\varphi_\alpha^{N,t}$ stand for the flow associated to the characteristics of the equation satisfied by μ^N. For any $0 \le t \le T$, we have $\varphi_0^{N,t}(B(0,M_\epsilon)) \subset B(0,A_\epsilon)$ so that $\complement\left(\varphi_t^{k,0}(B(0,A_\epsilon))\right) = \varphi_t^{N,0}\Big(\complement B(0,A_\epsilon)\Big) \subset \complement B(0,M_\epsilon)$. It follows that

$$\begin{aligned} \int_{\complement B(0,A_\epsilon)} \mathrm{d}\mu_t^N(x,v) &= \int_{\complement\varphi_t^{k,0}(B(0,A_\epsilon))} \mathrm{d}\mu_0^N(x,v) \\ &\le \int_{\complement B(0,M_\epsilon)} \mathrm{d}\mu_0^N(x,v) \le \epsilon. \end{aligned}$$

By a standard approximation, we check that the same estimate is satisfied by the limit μ. Finally, since the tight convergence is equivalent to the convergence with respect to the Kantorovich-Rubinstein distance W_1, we conclude that

$$\lim_{N\to\infty}\Big(\sup_{0\le t\le T} W_1(\mu_t^N,\mu_t)\Big) = 0$$

According to Lemma 4.3.3 and Lemma 4.4.4, the following mapping

$$\begin{aligned} \mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d) \times C_{W_1}\Big([0,T],\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\Big) &\longrightarrow C_{W_1}\Big([0,T],\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\Big) \\ (\mu_0,\mu) &\longmapsto \mathcal{T}_{\mu_0}(\mu) \end{aligned}$$

is continuous. Then, we get

$$\mu^N = \mathcal{T}_{\mu_0^N}(\mu^N) \xrightarrow[N\to\infty]{C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))} \mathcal{T}_{\mu_0}(\mu).$$

It implies that $\mu = \mathcal{T}_{\mu_0}(\mu)$ and $\mu \in C_{W_1}\Big([0,T],\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d\Big)$ satisfies (4.19), which ends the proof. ∎

4.5 Mean Field Limit for the Vlasov–Wave–Fokker-Planck system

Preliminary observations

In this section, we consider the case where the Fokker-Planck operator is added in the kinetic equation: namely the equation for the particle distribution in (4.6) is replaced by (4.9). We shall establish that this system can be obtained as the limit $N \to \infty$ from the system of stochastic differential equations (4.11). We remark that the right hand side of the wave equation in (4.11) is nothing but

$$-\sigma_2(y)\sigma_1 \star \widehat{\rho}_t^N(x)$$

with $\widehat{\rho}_t^N(x) = \int_{\mathbb{R}^d} \mathrm{d}\mu_t^N(x,v)$ and $\widehat{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{(q_j(t),p_j(t))}$ is the empirical measure associated to (4.11). We then use Lemma 4.3.2 again to recast (4.11) as follows

$$\left\{ \begin{array}{l} \mathrm{d}q_i^N(t) = p_i^N(t)\,\mathrm{d}t \\ \mathrm{d}p_i^N(t) = -\nabla_x(V + \Phi_0 - \mathcal{L}(\widehat{\mu}^N))(t, q_i^N(t))\,\mathrm{d}t - \gamma p_i^N(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B_i(t), \end{array} \right. \tag{4.32}$$

for any $i \in \{1, ..., N\}$, where from now on we emphasize the dependence with respect to N. We also remind the reader that the $(B_j(t))_{t\geq 0}$'s are independent Brownian motions. In this context the family of positions and velocities $t \mapsto (q_j^N(t), p_j^N(t))_{j\in\{1,...,N\}}$ is made of random variables depending on the Brownian motions and on the initial data which can be random (independently distributed) too. Indeed, the initial positions and velocities in (4.12) are supposed to be distributed according to

$$\mathbb{P}[(q_{0,j}^N, p_{0,j}^N) \in A] = \int_A \mathrm{d}\mu_0 \quad \text{for any } j \in \{1, ..., N\}. \tag{4.33}$$

In contrast to what happened for (4.3), here the empirical distribution $\widehat{\mu}^N$ is no longer a solution of (4.9). There are several arguments to convince ourselves of this fact [11]:

- due to the diffusion operator (with respect to velocity) in (4.9), we cannot expect that the solution of the kinetic equation remains a sum of Dirac masses for positive times,
- by nature $\widehat{\mu}^N$ is a random variable (due to the Brownian motions) while the solution of (4.9) is a deterministic quantity (at least when we work with deterministic initial data).

Actually, it is possible to compute the equation satisfied by $\widehat{\mu}^N$. Let us use the shorthand notation $z_j^N(t) = (q_j^N(t), p_j^N(t))$ to specify the solution of (4.11). Applying Itô's formula to integrate (4.11), we get

$$\begin{aligned} \varphi(z_j^N(t)) - \varphi(z_j^N(0)) = &\int_0^t \nabla_x\varphi(z_j^N(s)) \cdot p_j^N(s)\,\mathrm{d}s \\ &- \int_0^t \nabla_v\varphi(z_j^N(s)) \cdot \Big(\nabla_x(V + \Phi_0(s) - \mathcal{L}(\widehat{\mu}^N)(s))(q_j^N(s)) + \gamma p_j^N(s)\Big)\,\mathrm{d}s \\ &+\gamma \int_0^t \Delta_v\varphi(z(s))\,\mathrm{d}s + \sqrt{2\gamma}\int_0^t \nabla_v\varphi(z_j^N(s)) \cdot \,\mathrm{d}B_j(s) \end{aligned}$$

for any trial function $\varphi \in C(\mathbb{R}^d \times \mathbb{R}^d)$ and any $j \in \{1, ..., N\}$. Let us average over $j \in \{1, ..., N\}$. We obtain the following weak relation satisfied by $\widehat{\mu}^N$

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z) \ \mathrm{d}\widehat{\mu}_t^N(z) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\widehat{\mu}_0^N(z) = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\varphi(z)\cdot v\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s \\ - \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot \Big(\nabla_x(V+\Phi_0(s) - \mathcal{L}(\widehat{\mu}^N)(s))(x) + \gamma v\Big)\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s \\ +\gamma\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \Delta_v\varphi(z)\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s + \mathcal{I}_N,\end{aligned} \tag{4.34}$$

where $\mathcal{I}_N$ is defined by

$$\mathcal{I}_N = \frac{\sqrt{2\gamma}}{N}\sum_{j=1}^N \int_0^t \nabla_v\varphi(z_j^N(s))\cdot\ \mathrm{d}B_j(s). \tag{4.35}$$

In general $\mathcal{I}_N \neq 0$ cannot be expressed by means of $\widehat{\mu}^N$, and the equation is not closed. However, we shall see that $\mathbb{E}[|\mathcal{I}_N|]$ is of order $\mathscr{O}(\frac{1}{\sqrt{N}})$; accordingly $\widehat{\mu}^N$ tends to be a solution of (4.9). Moreover, the martingale theory ensures that

$$\mathbb{E}[\mathcal{I}_N] = 0.$$

Indeed, since the Brownian motion is a martingale, $\mathcal{I}_N$ is a martingale too (see [53, Definition 2.9 Chapter 3]), and its expectation value does not depend on t and we conclude immediately since at $t = 0$ we have $\mathcal{I}_N = 0$. This observation motivates to introduce the measure $\mu^{(1,N)}$ defined by the following identity

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\mu_t^{(1,N)}(z) = \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\widehat{\mu}_t^N\right]. \tag{4.36}$$

This measure can be related to the particle distribution in the $N-$body phase space: $\mu_t^{(N)}$ lies in $\mathcal{M}^1((\mathbb{R}^d \times \mathbb{R}^d)^N)$ and it is defined by

$$\mu_t^{(N)}(\mathscr{A}) = \mathbb{P}[(z_1^N(t), ..., z_N^N(t)) \in \mathscr{A}]$$

for any $\mathscr{A} \subset (\mathbb{R}^d \times \mathbb{R}^d)^N$. Since (4.11) and (4.33) do not change if we permute the particles $\{z_1^N, .., z_N^N\}$, all the random variables $z_j(t)$ share the same probability $\tilde{\mu}_t^{(1,N)}$, which is nothing but the first marginal of $\mu_t^{(N)}$: for any $A \subset \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\tilde{\mu}_t^{(1,N)}(A) = \int_{A\times(\mathbb{R}^d\times\mathbb{R}^d)^{N-1}} \ \mathrm{d}\mu^{(N)}(z) = \mathbb{P}[z_1^N(t) \in A].$$

We go back to (4.36) by the following simple computation: for any $A \subset \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\mu_t^{(1,N)}(A) = \mathbb{E}[\widehat{\mu}_t^N(A)] = \frac{1}{N}\sum_{i=1}^N \mathbb{P}[z_i^N(t) \in A] = \frac{1}{N}\sum_{i=1}^N \mathbb{P}[z_1^N(t) \in A] = \tilde{\mu}_t^{(1,N)}(A).$$

We can thus identify the two measures $\mu_t^{(1,N)} = \tilde{\mu}_t^{(1,N)} \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$.

In (4.34), bearing in mind (4.16), we can write

$$\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\nabla_x\mathcal{L}(\widehat{\mu}^N)(s)(x)\,\mathrm{d}\widehat{\mu}^N_s(z)\,\mathrm{d}s$$

$$= \int_0^t \int_0^s p(s-\tau) \sum_{1\leq i,j\leq N} \nabla_v\varphi(z_i^N(s))\cdot\nabla_x\Sigma(q_i^N(s)-q_j^N(\tau))\,\mathrm{d}\tau\,\mathrm{d}s,$$

We take the expectation value in (4.34). We are led to the following weak equation satisfied by $\mu^{(1,N)}$

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(z)\,\mathrm{d}\mu_t^{(1,N)}(z) - \int_{\mathbb{R}^d\times\mathbb{R}^d}\varphi(z)d\mu_0^{(1,N)}(z) = \int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_x\varphi(z)\cdot v\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s \quad (4.37)\\ -\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla_v\varphi(z)\cdot\Big(\nabla_x(V+\Phi_0(s))(x)+\gamma v\Big)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s\\ +\int_0^t\int_0^s\int_{(\mathbb{R}^d\times\mathbb{R}^d)^2}p(s-\tau)\\ \times\nabla\Sigma(x_1-x_2)\cdot\nabla_v\varphi(z_1)\,\mathrm{d}\left(\frac{N-1}{N}\mu_{s,\tau}^{(2,N)}+\frac{1}{N}\nu_{s,\tau}^{(2,N)}\right)(z_1,z_2)\,\mathrm{d}\tau\,\mathrm{d}s\\ +\gamma\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\Delta_v\varphi(z)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s+0,\end{aligned}$$

where

- $\mu_{s,\tau}^{(2,N)}$ is the joint probability measure of two different particles at different times s and τ: for $B\subset(\mathbb{R}^d\times\mathbb{R}^d)^2$, we have
$$\mu_{s,\tau}^{(2,N)}(B) = \mathbb{P}\Big((z_1^N(s), z_2^N(\tau))\in B\Big), \tag{4.38}$$
- $\nu_{s,\tau}^{(2)}$ is the joint probability measure of one particle at two different times s and τ: for $B\subset(\mathbb{R}^d\times\mathbb{R}^d)^2$, we have
$$\nu_{s,\tau}^{(2,N)}(B) = \mathbb{P}\Big((z_1^N(s), z_1^N(\tau))\in B\Big). \tag{4.39}$$

Equation (4.37) is still not closed, due to the joint probability measures $\mu_{s,\tau}^{(2,N)}$ and $\nu_{s,\tau}^{(2,N)}$. Besides, we can write similarly the equations satisfied by $\mu^{(2,N)}$ or $\nu^{(2,N)}$, which imply the third order probability measures and so on... This BBGKY hierarchy is non standard because of the coupling with the wave equation which induces a half-convolution in time. Here, we focus on the behavior of (4.37) as $N\to\infty$. Since $\nu^{(2,N)}$ is a probability measure, it is clear that the corresponding term in (4.37) is of order $\mathscr{O}(1/N)$, and thus it goes to 0 as $N\to\infty$. The difficulties relies on the terms with $\mu^{(2,N)}$. Initially, the N particles are independent; this property is not conserved for positive times, but we expect that particles tend to be less and less correlated as N becomes large, which amounts to say that $\mu_{s,\tau}^{(2,N)}$ looks like the product $\mu_s^{(1,N)}\otimes\mu_\tau^{(1,N)}$. We shall make this intuition rigorous, which eventually justifies that $\mu^{(1,N)}$ tends to a solution of (4.9). As said above, new difficulties are related to the unusual half–convolution with respect to the time variable in the interaction operator. To handle this, we shall introduce a suitable notion of "multi-times propagation of chaos", which is inspired from the following Definition [80], see also [43, 70, 71]

Definition 4.5.1 *a) Let E be a separable metric space and let $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ be a sequence of symmetric probability measures on E^N. Let μ be a probability measure on E. We say that $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ is* μ-chaotic *if for any $k\in\mathbb{N}\setminus\{0\}$ and any $\varphi_1,..,\varphi_k$ in $C\cap L^\infty(E)$ the following identity holds*

$$\lim_{N\to\infty}\int_{E^N}\varphi_1(z_1)...\varphi_k(z_k)\,\mathrm{d}\mu^N(z_1,..,z_N)=\prod_{i=1}^k\int_E\varphi_i(z)\,\mathrm{d}\mu(z). \tag{4.40}$$

b) We say that a Markov process leading the evolution of a family $\Big\{\mu^N : t\in[0,T]\mapsto \mu_t^N\in\mathcal{M}^1(E^N),\ N\in\mathbb{N}\setminus\{0\}\Big\}$ of probability measures on E^N propagates the chaos if, given a sequence $(\mu_0^N)_{N\in\mathbb{N}\setminus\{0\}}$ of μ-chaotic initial data, the sequence $(\mu_t^N)_{N\in\mathbb{N}}$ is also $\mu-$chaotic for all $t>0$.

Applying Definition 4.5.1-a) with $k=1$ in (4.40), we see that μ is the weak limit in $\mathcal{M}^1(E)$ of the first marginal $\mu^{(1,N)}$ of μ^N as $N\to\infty$. In (4.33), we made a strong, but natural, assumption on the initial data, namely it factorizes: $\mu_0^N=\mu_0^{\otimes N}$. However, due to the interactions with the medium, this property has no reason to be preserved for positive times. The assumption for the initial data based on (4.40) is far weaker, and it is well–adapted to our purposes since it is preserved by the dynamics, as we shall see below in Corollary 4.5.7-ii) . Finally, to any sequence $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ of probability measures on E^N, we associate the family $(\widehat{\mu}^N)_{N\in\mathbb{N}\setminus\{0\}}$ of random measures on E defined by

$$\widehat{\mu}^N=\frac{1}{N}\sum_{i=1}^N\delta_{z_i}$$

where the random variable $(z_1,..,z_N)$ is distributed in E^N according to μ^N. The following result due to [80] makes the connection between the empirical distribution and the first marginal.

Proposition 4.5.2 *A sequence $(\mu^N)_{N\in\mathbb{N}}$ is μ-chaotic iff (4.40) is satisfied for $k=2$. Equivalently, $\widehat{\mu}^N$ converges in law to μ, the weak limit in $\mathcal{M}^1(E)$ of the first marginal $\mu^{(1,N)}$, as $N\to\infty$.*

We explain now how we will proceed, following the arguments discussed in [80]. Assume that we have at hand a measure-valued solution $\mu\in C_{W_1}([0,\infty);\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$ of (4.9). We introduce the following system of stochastic differential equations

$$\begin{cases}\mathrm{d}\tilde{q}_i^N=\tilde{p}_i^N(t)\,\mathrm{d}t,\\ \mathrm{d}\tilde{p}_i^N=-\nabla_x(V+\Phi_0-\mathcal{L}(\mu))(t,\tilde{q}_i^N(t))\,\mathrm{d}t-\gamma\tilde{p}_i^N(t)\,\mathrm{d}t+\sqrt{2\gamma}\,\mathrm{d}B_i(t),\end{cases} \tag{4.41}$$

for $i\in\{1,...,N\}$, where the Brownian motion $(B_i(t))_{t>0}$ are the same as in (4.11). The initial data for (4.41)

$$\tilde{q}_i^N(0)=q_{0,i}^N,\qquad \tilde{p}_i^N(0)=p_{0,i}^N \tag{4.42}$$

are also shared with (4.11). We suppose that (4.33) is fulfilled. The dynamics of these "fictitious" particles is driven by the measure μ. We are going to prove the following result, which shows that this dynamics is close to those of the original system.

Theorem 4.5.3 *Let* $(z_i^N)_{j\in\{1,\dots,N\}} = (q_i^N, p_i^N)_{j\in\{1,\dots,N\}}$ *be a solution of* (4.11)*, with initial data given by* (4.33)*. Let* $(\tilde z_i^N)_{j\in\{1,\dots,N\}} = (\tilde q_i^N, \tilde p_i^N)_{j\in\{1,\dots,N\}}$ *be a solution of* (4.41) *with the same initial data. Let* $0 < T < \infty$*. We can find a constant* C_T *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |z_i^N - \tilde z_i^N|(t)\right] \le \frac{C_T}{\sqrt{N}}.$$

We will deduce several consequences from this statement:

- it implies the convergence of the first marginal $\mu^{(1,N)}$ to μ for a certain Kantorowich–Rubinstein distance, see Corollary 4.5.7-i),
- the convergence in law of the empirical measure $\widehat{\mu}^N$ to the same limit then follows from Proposition 4.5.2,
- it allows us to establish the propagation of chaos for the solution of (4.11), see Corollary 4.5.7-ii),
- and, coming back to (4.34), (4.35) and (4.37), it allows us to prove that $\mathbb{E}[|\mathcal{I}_N|]$ goes to 0 as $N \to \infty$ (see lemma 4.5.8) while
$$\int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\nabla\sigma(x_1 - x_2)\cdot\nabla_v\varphi(z_1)\,\mathrm{d}\mu_{s,\tau}^{(2,N)}(z_1,z_2)\,\mathrm{d}\tau\,\mathrm{d}s$$
behaves like
$$\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\nabla_x\mathcal{L}(\mu^{(1,N)})(s,x)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s$$
as expected since both converge to
$$\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\nabla_x\mathcal{L}(\mu)(s,x)\,\mathrm{d}\mu_s(z)\,\mathrm{d}s.$$

These results justify (4.9) as the equation satisfied by the limit of the first marginal $\mu^{(1,N)}$ and the empirical distribution $\widehat{\mu}^N$ associated to (4.11) (which tend to coincide for a large number of particles) when N goes to infinity. This Section is organized as follows. Firstly, we prove the existence and uniqueness of the solution μ of (4.9) and of the random variables $(z_i^N)_{i\in\{1,\dots,N\}}$ and $(\tilde z_i^N)_{i\in\{1,\dots,N\}}$. Secondly, we establish Theorem 4.5.3 and its consequences.

Analysis of the stochastic equations and the PDE system

N–particles system

We first prove that the system (4.11) is well posed for data that verify (4.33).

Theorem 4.5.4 *The system* (4.11) *with* (4.33) *has a unique strong solution in the sense of the stochastic differential equation. It means that, fixing a family of Brownian motions* $(B_i)_{i\in\{1,\dots,N\}}$ *and a family of initial data* $(q_{0,i}^N, p_{0,i}^N)_{i\in\{1,\dots,N\}}$ *there exists a unique continuous family* $(t\mapsto(q_i^N(t), p_i^N(t)))_{i\in\{1,\dots,N\}}$ *solution of* (4.11) *with* (4.33).

The proof is just an adaptation of the Cauchy–Lipschitz theorem. In order to simplify the forthcoming computations, let us set, for $\mu \in C_{W_1}([0,T];\mathbb{R}^d \times \mathbb{R}^d)$,

$$F(\mu)(t,q) = -\nabla_x(V + \Phi_0 - \mathcal{L}(\mu))(t,q).$$

The following estimate on F will be useful for the analysis.

Lemma 4.5.5 *Let μ_1, μ_2 be to probability measures on $C_{W_1}([0,T];\mathbb{R}^d \times \mathbb{R}^d)$. Let $z_1 = (q_1, p_1)$ and $z_2 = (q_2, p_2)$. We set*

$$c_1(t) = \|\nabla^2 V\|_{L^\infty} + \|\nabla^2 \sigma_1\|_{L^2}\|\sigma_2\|_{L^2}\left(\|\Psi_0\|_{L^2} + t\|\Psi_1\|_{L^2}\right) + \|\sigma_2\|_{L^2}^2\|\nabla\sigma_1\|_{L^2}^2\frac{t^2}{2}.$$

Then, we have:

$$|F(\mu_1)(t,q_1) - F(\mu_2)(t,q_2)| \leq c_1(t)|z_1 - z_2| + |\nabla\mathcal{L}(\mu_1 - \mu_2)(t,q_1)|.$$

Proof. Lemma 4.3.3 allows us to obtain the following estimate

$$\begin{aligned}
|(-\nabla_x(V + \Phi_0 & -\mathcal{L}(\mu_1))(t,q_1)) - (-\nabla_x(V + \Phi_0 - \mathcal{L}(\mu_2))(t,q_2)| \\
& \leq (\|\nabla^2 V\|_{L^\infty(\mathbb{R}^d)} + \|\nabla^2\Phi_0(t)\|_{L^\infty(\mathbb{R}^d)})|q_1 - q_2| + \\
& \quad + \|\nabla^2\mathcal{L}(\mu_1)(t)\|_{L^\infty(\mathbb{R}^d)}|q_1 - q_2| + \|\nabla\mathcal{L}(\mu_1 - \mu_2)(t)\|_{L^\infty(\mathbb{R}^d)} \\
& \leq c_1(t)|q_1 - q_2| + |\nabla\mathcal{L}(\mu_1 - \mu_2)(t,q_1)|.
\end{aligned}$$

■

Proof of Theorem 4.5.4. Let us introduce a few shorthand notations. We define $Z^N \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ by

$$Z^N = (q_1, p_1, ..., q_N, p_N).$$

Next we introduce the force field

$$F^N(\widehat{\mu}^N)(t,Z^N) = \begin{pmatrix} p_1 \\ F(\widehat{\mu}^N)(t,q_1) - \gamma p_1 \\ \vdots \\ p_N \\ F(\widehat{\mu}^N)(t,q_N) - \gamma p_N \end{pmatrix}.$$

We define the diffusion matrix Γ^N, which lies in $M_{2Nd}(\mathbb{R})$, by

$$\Gamma^N = \begin{pmatrix} 0 & & & & \\ & \sqrt{2\gamma}\mathrm{Id}_{\mathbb{R}^d} & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \sqrt{2\gamma}\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}.$$

Finally, since the family of Brownian motions $(B_i(t))_{t\geq 0}$ in $C([0,\infty);\mathbb{R}^d \times \mathbb{R}^d)$ can be described by the a single Brownian motion $(B^N(t))_{t\geq 0}$ in $C([0,\infty);(\mathbb{R}^d \times \mathbb{R}^d)^N)$, the system

(4.11) can be recast as

$$\left\{\begin{array}{l} \mathrm{d}Z^N = F^N(\widehat{\mu}^N_{Z^N})(t,Z^N)\,\mathrm{d}t + \Gamma^N\,\mathrm{d}B^N_t \\ Z^N(0) = Z^N_0 \end{array}\right. \tag{4.43}$$

with

$$\widehat{\mu}^N_{Z^N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{(Z^N_{2i-1}(t),Z^N_{2i}(t))}.$$

We now fix the Brownian motion $(B^N(t))_{t\geq 0}$ and the initial data Z^N_0 verifying (4.33), and we are going to prove that (4.43) has a unique solution.

For any continuous function Z in $C([0,\infty);(\mathbb{R}^d\times\mathbb{R}^d)^N)$, we set

$$\mathcal{T}(Z)(t) = Z_0 + \int_0^t F^N(\widehat{\mu}^N_Z)(s,Z(s))\,\mathrm{d}s + \Gamma^N B^N_t.$$

Let Z^1 and Z^2 be in $C([0,\infty);(\mathbb{R}^d\times\mathbb{R}^d)^N)$. By using Lemma 4.3.3 we obtain

$$\begin{aligned} |\nabla\mathcal{L}(\widehat{\mu}^N_{Z^1}-\widehat{\mu}^N_{Z^2})(t,x)| &\leq \int_0^t \frac{|p(t-s)|}{N}\sum_{i=1}^N |\nabla\Sigma(x-q^1_i(s)) - \nabla\Sigma(x-q^2_i(s))|\,\mathrm{d}s \qquad (4.44)\\ &\leq \|\nabla^2\Sigma\|_{L^\infty(\mathbb{R}^d)}\frac{1}{N}\sum_{i=1}^N\int_0^t |p(t-s)||q^2_i(s)-q^1_i(s)|\,\mathrm{d}s\\ &\leq \|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\int_0^t (t-s)|Z^1(s)-Z^2(s)|\,\mathrm{d}s\\ &\leq \|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\frac{t^2}{2}\|Z^1-Z^2\|_{L^\infty(0,t)}. \end{aligned}$$

Let us set $c_2 = \frac{1}{2}\|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}$. We now make use of Lemma 4.5.5 in order to obtain the estimate

$$\begin{aligned} |\mathcal{T}(Z^1)-\mathcal{T}(Z^2)|(t) &= \left|\int_0^t \left(F(\widehat{\mu}^N_{Z^1})(s,Z^1(s)) - F(\widehat{\mu}^N_{Z^2})(s,Z^2(s))\right)\mathrm{d}s\right|\\ &\leq \int_0^t ((1+\gamma+c_1(s))|Z^1(s)-Z^2(s)| + \|\nabla\mathcal{L}(\widehat{\mu}^N_{Z^1}-\widehat{\mu}^N_{Z^2})(s)\|_{L^\infty})\,\mathrm{d}s\\ &\leq \int_0^t (1+\gamma+c_1(s)+c_2s^2)\|Z^1-Z^2\|_{L^\infty(0,s)}\,\mathrm{d}s. \end{aligned}$$

Let $C_T = 1+\gamma+c_1(T)+c_2T^2$. We get, for all $0\leq t\leq T<\infty$,

$$\|\mathcal{T}(Z^1)-\mathcal{T}(Z^2)\|_{L^\infty(0,t)} \leq C_T\int_0^t \|Z^1-Z^2\|_{L^\infty(0,s)}\,\mathrm{d}s.$$

By induction, we deduce that

$$\|\mathcal{T}^\ell(Z^1)-\mathcal{T}^\ell(Z^2)\|_{L^\infty(0,t)} \leq \frac{(tC_T)^\ell}{\ell!}\|Z^1-Z^2\|_{L^\infty(0,T)}$$

holds for any $\ell\in\mathbb{N}$ and any $0\leq t\leq T$. It shows that for ℓ such that $\frac{(tC_T)^\ell}{\ell!}<1$, $\mathcal{T}^\ell$ is a contraction. Therefore, there exists a unique fixed point in $C([0,T];(\mathbb{R}^d\times\mathbb{R}^d)^N)$ for any $T>0$, which ends the proof. ■

Vlasov–Wave–Fokker–Planck system

We now turn to prove the existence of solution for (4.9). To this end, we adopt a particle viewpoint by introducing the following system of stochastic differential equations

$$\begin{cases} \mathrm{d}x = v(t)\,\mathrm{d}t, \\ \mathrm{d}v = -\nabla_x(V + \Phi_0(t) - \mathcal{L}(\mu)(t))(x)\,\mathrm{d}t - \gamma v(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B(t), \\ \mu_t(A) = \mathbb{P}[(x(t), v(t)) \in A], \\ \mu(0) = \mu_0, \end{cases} \tag{4.45}$$

which is now non linear, in contrast to (4.41), since the trajectories depends on μ, their probability measure. In fact, this is nothing but a different viewpoint on (4.9). Indeed, if $z = (x, v)$ is a solution of (4.45), Itô's formula yields

$$\begin{aligned} \varphi(z(t)) - \varphi(z(0)) = &\int_0^t (\nabla_x\varphi(z(s)) \cdot v(s) - \nabla_v\varphi(z(s)) \cdot F(\mu)(s, z(s)))\,\mathrm{d}s \\ &+\sqrt{2\gamma}\int_0^t \nabla_v\varphi(z(s))\,\mathrm{d}B(s) + \gamma\int_0^t \Delta_v\varphi(z(s))\,\mathrm{d}s, \end{aligned}$$

and by taking the expectation value, we get

$$\begin{aligned} &\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\ \ \mathrm{d}\mu_t(z) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\mu_0(z) \\ &\qquad = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\varphi(z) \cdot v - \nabla_v\varphi(z) \cdot F(\mu)(s, z))\,\mathrm{d}\mu_s(z)\,\mathrm{d}s \\ &\qquad\quad -\gamma\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} (v \cdot \nabla_v\varphi(z) + \Delta\varphi)\,\mathrm{d}\mu_s(z)\,\mathrm{d}s. \end{aligned}$$

It corresponds to the weak formulation of (4.9) (which is the analog of Lemma 4.4.1 in the stochastic framework). Based on this, we shall prove the following statement.

Theorem 4.5.6 *i) For any initial data μ_0 in $\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$, there exists a unique solution μ of (4.9) in $C_{W_1}([0, +\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$,*

ii) the process (x, v) solution of (4.45) is well defined.

For analyzing (4.45), we need to go back to the definition of the Kantorowitch–Rubinstein distance in Section 4.3. It is convenient to change the framework and to work with measures defined on the space $S = C([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ endowed with the distance $d(f, g) = \|f - g\|_{L^\infty(0,T)} \wedge 1$. We specify the corresponding Kantorowich–Rubinstein distance as D_T, namely

$$D_T(\mu, \nu) = \inf_\pi \Big\{ \int_{(C([0,T];\mathbb{R}^d\times\mathbb{R}^d))^2} (||f - g||_{L^\infty([0,T])} \wedge 1)\,\mathrm{d}\pi(f, g)\Big\}. \tag{4.46}$$

The interpretation in terms of tight convergence or strong convergence in the dual of Lipschitz functions remains true in that case but it is far more difficult to see concretely. Measurable sets of $S = C([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ are the elements of the $\sigma-$algebra generated by the sets of the form

$$B_{t,A} = \Big\{\phi(t) \in A \quad \text{for } \phi \in C([0, T]; \mathbb{R}^d \times \mathbb{R}^d)\Big\},$$

where t spans $[0,T]$ and A spans the set of Borel-sets in $\mathbb{R}^d\times\mathbb{R}^d$. Given a measure μ on S, we can define a measure–valued function, that we still denote $\mu : t\in[0,T]\mapsto \mu_t\in\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$ by the relation

$$\mu_t(A)=\mu(B_{t,A}).$$

Looking at a process Z with probability μ in $\mathcal{M}^1(S)$, Z is almost surely continuous by definition of μ. Them, the dominated convergence theorem allows us to deduce

$$W_1(\mu_{t_1},\mu_{t_2})=\inf_{X,Y}\mathbb{E}[|X-Y|\wedge 1]\le\inf_{Z}\mathbb{E}[|Z(t_1)-Z(t_2)|\wedge 1]\xrightarrow[t_1\to t_2]{}0$$

where the second infimum is taken over all the processes Z with probability μ. Accordingly, $\mathcal{M}^1(S)$ embeds in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. We can easily check that the distance D_T is stronger than the one we used previously on that set owing to the inequality

$$\sup_{0\le t\le T}W_1(\mu_t,\nu_t)\le D_T(\mu,\nu).$$

Reminding of (4.15), the following estimate

$$\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathrm{d}(\mu_s-\nu_s)\right|\le\|f\|_{\mathrm{Lip}}D_t(\mu,\nu),\tag{4.47}$$

holds for any $f\in W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ and $0\le s\le t$.

Proof of Theorem 4.5.6. The proof is based on a fixed point argument, again. Let $0<T<\infty$ and set $S=C([0,T];\mathbb{R}^d\times\mathbb{R}^d)$. We shall use the Kantorowich–Rubinstein distance D_T defined by (4.46). Let μ be a finite measure on S. We consider the following system

$$\begin{cases}\mathrm{d}x=v(t)\,\mathrm{d}t,\\ \mathrm{d}v=-\nabla_x(V+\Phi_0-\mathcal{L}(\mu)(t,x)\,\mathrm{d}t-\gamma v(t)\,\mathrm{d}t+\sqrt{2\gamma}\,\mathrm{d}B(t),\\ \mathbb{P}[(x_0,v_0)\in A]=\mu_0(A).\end{cases}\tag{4.48}$$

Since the field $(t,x,v)\mapsto(v,\nabla_x(V+\Phi_0-\mathcal{L}(\mu)(t,x)$ is continuous with respect to time and Lipschitz with respect to the phase space variable (x,v), the solution of (4.48) is a well defined continuous process (see [53, Chapter 5, Theorem 2.9]) . We introduce the mapping

$$A\subset\mathbb{R}^d\times\mathbb{R}^d\longmapsto\mathbb{P}\Big[(x(t),v(t))\in A\quad\text{where }(x,v)\text{ satisfies (4.48) }\Big],$$

which, in turn, defines a new probability measure on S that we denote $\mathscr{T}(\mu)$. Pick μ_1 and μ_2 in $\mathcal{M}^1(S)$. We denote $z_1=(x_1,v_1)$ and $z_2=(x_2,v_2)$ the processes associated to $\mathscr{T}(\mu_1)$ and $\mathscr{T}(\mu_2)$, respectively. We bear in mind that both the Brownian motion B and the initial data z_0 are fixed. Integrating (4.48) we get

$$\begin{cases}(x_1-x_2)(t)=\displaystyle\int_0^t(v_1-v_2)(s)\,\mathrm{d}s,\\ (v_1-v_2)(t)=\displaystyle\int_0^t(F(\mu_1)(s,z_1(s))-F(\mu_2,(s,z_2(s))-\gamma(v_1-v_2)(s))\,\mathrm{d}s.\end{cases}$$

We deduce that both x_1-x_2 and v_1-v_2 are derivable. Using Lemma 4.5.5, we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(z_1-z_2)(t)\right|\le(1+\gamma+c_1(t))|z_1-z_2|(t)+\|\nabla_x\mathcal{L}(\mu_1-\mu_2)(t)\|_{L^\infty(\mathbb{R}^d)}.$$

Applying the Grönwall lemma, we get

$$\begin{aligned}|z_1 - z_2|(t) &\le \int_0^t e^{\int_s^t (1+\gamma+c_1(s))\,\mathrm{d}s} \|\nabla_x \mathcal{L}(\mu_1-\mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s\\ &\le C_T \int_0^t \|\nabla_x \mathcal{L}(\mu_1-\mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s,\end{aligned}$$

for all $0 \le t \le T$, where

$$C_T := \exp\left(\int_0^T (1+\gamma+c_1(s))\,\mathrm{d}s\right).$$

Notice that the right hand side is a deterministic monotically increasing function of t. Hence, we deduce that

$$D_t(\mathscr{T}(\mu_1), \mathscr{T}(\mu_2)) = \inf_{z_1,z_2} \mathbb{E}\left[||z_1 - z_2||_{L^\infty([0,t])} \wedge 1\right] \le C_T \int_0^t \|\nabla_x \mathcal{L}(\mu_1-\mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s \tag{4.49}$$

holds. We now turn to dominate the right hand side by $D_t(\mu_1,\mu_2)$. Let $x \in \mathbb{R}^d$ and $0 \le s \le t$; owing to (4.47) and Lemma 4.3.3, we have

$$\begin{aligned}|\nabla_x \mathcal{L}\ (\mu_1-\mu_2)(s,x)| &= \left|\int_0^s p(s-\tau)\int_{\mathbb{R}^d\times\mathbb{R}^d} \Sigma(x-y)\,\mathrm{d}(\mu_{1,\tau}-\mu_{2,\tau})(y,v)\,\mathrm{d}\tau\right|\,\mathrm{d}\tau\\ &\le \int_0^s |p(s-\tau)|\Big(2\|\Sigma(x-\cdot)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla(\Sigma(x-\cdot)\|_{L^\infty(\mathbb{R}^d)}\Big) D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau\\ &\le \|\sigma_2\|_{L^2}^2\|\sigma_1\|_{L^2}(2\|\sigma_1\|_{L^2} + \|\nabla\sigma_1\|_{L^2})\int_0^s (s-\tau) D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau.\end{aligned}$$

Setting $c_3 = \|\sigma_2\|_{L^2}^2\|\sigma_1\|_{L^2}(2\|\sigma_1\|_{L^2} + \|\nabla\sigma_1\|_{L^2})$, we obtain

$$\|\nabla_x \mathcal{L}(\mu_1-\mu_2)(s)\|_{L^\infty(\mathbb{R}^d)} \le c_3 T \int_0^s D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau.$$

Finally, inserting this estimate into (4.49), we arrive at

$$D_t(\mathscr{T}(\mu_1), \mathscr{T}(\mu_2)) \le c_3 T C_T \int_0^t\int_0^s D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau\,\mathrm{d}s.$$

We deduce by induction that

$$D_t(\mathscr{T}^\ell(\mu_1), \mathscr{T}^\ell(\mu_2)) \le \frac{(c_3 T C_T t)^{2\ell}}{(2\ell)!} D_T(\mu_1,\mu_2)$$

holds for any $\ell \in \mathbb{N}$ and any $0 \le t \le T$. It shows that for ℓ such that $\frac{(c_3 C_T T^2)^{2\ell}}{(2\ell)!} < 1$, $\mathscr{T}^\ell$ is a contraction for the distance D_T. Therefore, $\mathscr{T}$ has a unique fixed point in $\mathcal{M}^1(S)$, and thus $\mu_t \in C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$, for any $T>0$. It ends the proof. ∎

Asymptotic analysis

This Section is devoted to the proof of Theorem 4.5.3. The analysis relies on many estimates that have been established above. We compare $(z_i^N)_{i\in\{1,\dots,N\}}$, solution of (4.11) and $(\tilde{z}_i^N)_{i\in\{1,\dots,N\}}$, solution of (4.41). The two equations start from the same initial data $(z_{i,0}^N)_{i\in\{1,\dots,N\}}$ and involve the same family of Brownian motions $(B_i(t))_{i\in\{1,\dots,N\}}$. In fact, for any fixed $i\in\{1,\dots,N\}$, $t\mapsto(\tilde{q}_i^N(t),\tilde{p}_i^N(t)) = \tilde{z}_i^N(t)$ is the solution of (4.48) for the initial data $q_{0,i}^N, p_{0,i}^N$ distributed according to the common measure μ_0, see (4.33). In particular, we shall use the fact that

$$\mu_t(\,\mathrm{d}z) \text{ is the common law of the } (\tilde{q}_i^N(t), \tilde{p}_i^N(t))\text{'s.}$$

We estimate the difference

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(q_i^N - \tilde{q}_i^N)(t) = (p_i^N - \tilde{p}_i^N)(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(p_i^N - \tilde{p}_i^N)(t) = (-F(\widehat{\mu}^N)(t, q_i^N(t)) + F(\mu)(t, \tilde{q}_i^N(t))) - \gamma(p_i^N - \tilde{p}_i^N)(t). \end{cases}$$

Using Lemma 4.5.5, we get (at least in the sense of distributions)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|z_i^N - \tilde{z}_i^N\right|(t) \le (1 + \gamma + c_1(t))|z_i^N - \tilde{z}_i^N|(t) + |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(t, \tilde{q}_i^N(t))|.$$

Since the initial data coincide, we thus have

$$\left|z_i^N - \tilde{z}_i^N\right|(t) \le \int_0^t (1 + \gamma + c_1(s))|z_i^N - \tilde{z}_i^N|(s)\,\mathrm{d}s + \int_0^t |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s.$$

In order to deal with the last term, we introduce the empirical density $\tilde{\mu}^N$ associated to the family $(\tilde{z}_i^N)_{i\in\{1,\ldots,N\}}$ solution of (4.41)

$$\tilde{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{z}_i^N(t)}. \tag{4.50}$$

Then, using (4.44), we split

$$\begin{aligned} |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(s, \tilde{q}_i^N(s))| \;&\le\; |\nabla\mathcal{L}(\widehat{\mu}^N - \tilde{\mu}^N)(s, \tilde{q}_i^N(s))| + |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))| \\ &\le\; c_2\int_0^s \frac{s-\sigma}{N}\sum_{j=1}^N |z_j^N - \tilde{z}_j^N|(\sigma)\,\mathrm{d}\sigma + |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|. \end{aligned}$$

We are thus led to

$$\begin{aligned} \sup_{0\le\tau\le t}\left|z_i^N - \tilde{z}_i^N\right|(\tau) \;&\le\; \int_0^t (1 + \gamma + c_1(s))|z_i^N - \tilde{z}_i^N|(s)\,\mathrm{d}s \\ &\quad + \int_0^t\int_0^s c_2\frac{s-\sigma}{N}\sum_{j=1}^N |z_j^N - \tilde{z}_j^N|(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \\ &\quad + \int_0^t |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s. \end{aligned}$$

We shall take the expectation value in this inequality. As (4.11), with (4.33) and (4.41)-(4.42) do not change if we permute the indices $i \in \{1, .., N\}$ of the particles, all the random variables $|z_i^N - \tilde{z}_i^N|(t)$ share the same probability and the same expectation value. Therefore, we remark that, for any $i, j \in \{1, ..., N\}$ and $s \ge 0$, we have

$$\mathbb{E}\sup_{0\le\tau\le s}\left|z_i^N - \tilde{z}_i^N\right|(\tau) = \mathbb{E}\sup_{0\le\tau\le s}\left|z_j^N - \tilde{z}_j^N\right|(\tau) = \frac{1}{N}\sum_{k=1}^N \mathbb{E}\sup_{0\le\tau\le s}\left|z_k^N - \tilde{z}_k^N\right|(\tau).$$

It allows us to obtain

$$\begin{aligned}\mathbb{E}\sup_{0\le\tau\le t}\left|(z_i^N-\tilde z_i^N)(\tau)\right| &\le \int_0^t\left(1+\gamma+c_1(s)+c_2\frac{s^2}{2}\right)\mathbb{E}\sup_{0\le\sigma\le s}|z_i^N-\tilde z_i^N|(s)\,\mathrm{d}s\\ &\quad+\int_0^t\mathbb{E}|\nabla\mathcal{L}(\tilde\mu^N-\mu)(s,\tilde q_i^N(s))|\,\mathrm{d}s.\end{aligned}$$

Hence, by using the Grönwall lemma, we deduce that, for any $0<T<\infty$ we can find $C_T>0$ such that

$$\mathbb{E}\sup_{0\le\tau\le t}\left|z_i^N-\tilde z_i^N\right|(\tau)\le C_T\int_0^t\mathbb{E}|\nabla\mathcal{L}(\tilde\mu^N-\mu)(s,\tilde q_i^N(s))|\,\mathrm{d}s \tag{4.51}$$

holds for any $0\le t\le T$. We are going to estimate the right hand side as a consequence of the law of the large numbers. For any iid square integrable family of random variables $(Y_i)_{1\le i\le N}$ we indeed remind the reader that

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N Y_i-\mathbb{E}[Y_1]\right|\right]\le\frac{(\mathrm{Var}(Y_1))^{1/2}}{\sqrt{N}} \tag{4.52}$$

holds. We now show that $\mathbb{E}\left[|\nabla\mathcal{L}(\tilde\mu^N-\mu)(s,\tilde q_1^N(s))|\right]$ can be written in a form similar to the left hand side of (4.52). By definition of $\tilde\mu^N$ as the empirical measure of the $\tilde z_i^N$, see (4.50), we have

$$\mathbb{E}\left[|\nabla\mathcal{L}(\tilde\mu^N-\mu)(s,\tilde q_1^N(s))|\right]=\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N\nabla\mathcal{L}(\delta_{\tilde z_i^N})(s,\tilde q_1^N(s))-\nabla\mathcal{L}(\mu)(s,\tilde q_1^N(s))\right|\right]. \tag{4.53}$$

In order to simplify the computations, we set $X_i^1(s)=\nabla\mathcal{L}(\delta_{\tilde z_i^N})(s,\tilde q_1^N(t))$ for $i\in\{1,...,N\}$. Assuming that we know $\tilde z_1$, the family $(X_2^1,X_3^1,...,X_N^1)$ is made of iid random variables with the common expectation value

$$\begin{aligned}\mathbb{E}\left[\nabla\mathcal{L}(\delta_{\tilde z_2^N})(s,\tilde q_1^N(s))\Big|\tilde z_1\right] &=\mathbb{E}\left[\int_0^s p(s-\sigma)\nabla\Sigma(\tilde q_1^N(s)-\tilde q_2^N(\sigma))\,\mathrm{d}\sigma\Big|\,\tilde z_1\right]\\ &=\int_0^s p(s-\sigma)\mathbb{E}\left[\nabla\Sigma(\tilde q_1^N(s)-\tilde q_2^N(\sigma))|\tilde z_1\right]\,\mathrm{d}\sigma\\ &=\int_0^s p(s-\sigma)\int_{\mathbb{R}^d\times\mathbb{R}^d}\nabla\Sigma(\tilde q_1^N(s)-y)\,\mathrm{d}\mu_\sigma(y,v)\,\mathrm{d}\sigma\\ &=\nabla\mathcal{L}(\tilde\mu)(s,\tilde q_1^N(s)).\end{aligned}$$

This observation allows us to split (4.53) as follows

$$\begin{aligned}\mathbb{E}\left[|\nabla\mathcal{L}(\tilde\mu^N-\mu)(s,\tilde q_1^N(s))|\right] &\le\frac{1}{N}\mathbb{E}\left[|X_1^1(s)-\mathbb{E}[X_2^1(s)|\tilde z_1]|\right]\\ &\quad+\frac{N-1}{N}\mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^N X_i^1(s)-\mathbb{E}[X_2^1(s)|\tilde z_1]\right|\right].\end{aligned} \tag{4.54}$$

Since $(X_2^1, X_3^1, ..., X_N^1)$ are iid, we estimate the second term of the right hand side with (4.52)

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^{N} X_i^1 - \mathbb{E}[X_2^1|\tilde{z}_1]\right|\right] &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^{N} X_i^1 - \mathbb{E}[X_2^1|\tilde{z}_1]\right| \quad \middle| \tilde{z}_1\right] \mathrm{d}\mu(\tilde{z}_1) \\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{\sqrt{N-1}} (\mathrm{Var}(X_2^1|\tilde{z}_1))^{1/2} \,\mathrm{d}\mu(\tilde{z}_1).\end{aligned}$$

From the estimates on $\mathcal{L}$, we also have $|X_i^1(s)| \leq \|\sigma_1\|_{L^2}\|\nabla\sigma_1\|_{L^2}\|\sigma_2\|_{L^2}^2 T =: c_4 T$ for any $0 \leq s \leq T < \infty$. Coming back to (4.54), we get

$$\mathbb{E}\left[|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_1^N(s))|\right] \leq \frac{2c_4 T}{N} + \frac{c_4 T}{\sqrt{N}}$$

Eventually, we insert this in (4.51) and we conclude that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |z_i^N - \tilde{z}_i^N|(t)\right] \leq \frac{3c_4 T C_T}{\sqrt{N}}$$

holds. It finishes the proof of Theorem 4.5.3.

Let us detail a few relevant consequences of Theorem 4.5.3.

Corollary 4.5.7 *Let μ be the solution of* (4.9) *We have*

i) With D_T the Kantorowich-Rubinstein distance on $\mathcal{M}^1(C([0,T];\mathbb{R}^d\times\mathbb{R}^d))$, defined by (4.46)*, we have*

$$D_T(\mu^{(1,N)}, \mu) \leq \frac{C}{\sqrt{N}}.$$

ii) Multi-time propagation of chaos holds: it means that, taking $0 \leq \tau \leq s$ and φ_1, φ_2 in $C \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\begin{aligned}&\lim_{N\to\infty} \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi_1(z_1)\varphi_2(z_2)\,\mathrm{d}\mu_{s,\tau}^{(2,N)}(z_1,z_2) \\ &\qquad = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(z)\,\mathrm{d}\mu_s(z)\right)\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_2(z)\,\mathrm{d}\mu_\tau(z)\right).\end{aligned}$$

In particular $\widehat{\mu}^N$ converges in law to μ.

We remind the reader that the joint probability measure of two different particles at two times $\mu_{s,\tau}^{(2,N)}$ is defined by (4.38). This statement allows us to interpret (4.9) as the equation satisfied by the limit as $N \to \infty$ of both the first marginal $\mu^{(1,N)}$ and the empirical distribution $\widehat{\mu}^N$ for a system of particles $(q_i^N, p_i^N)_{i\in\{1,...,N\}}$ solution of (4.11) with (4.33).

Proof of Corollary 4.5.7. Item i) is a direct consequence of the definition of D_T and the estimate in Theorem 4.5.3. Let us discuss item ii). From the definition of W_1, we have

$$\begin{aligned} W_1(\mu^{(2)}_{s,\tau}, \mu_s \otimes \mu_\tau) &= \mathbb{E}\left[|(z_1^N(s), z_2^N(\tau)) - (\tilde{z}_1^N(s), \tilde{z}_2^N(\tau))| \wedge 1\right] \\ &\leq \mathbb{E}\left[|z_1^N(s) - \tilde{z}_1^N(s)| \wedge 1\right] + \mathbb{E}\left[|z_2^N(s) - \tilde{z}_2^N(s)| \wedge 1\right] \\ &\leq \frac{2C}{\sqrt{N}}. \end{aligned} \tag{4.55}$$

Since the Kantorovich-Rubinstein distance metrizes the tight convergence, we get for all φ in $C_b((\mathbb{R}^d \times \mathbb{R}^d)^2)$

$$\lim_{N\to\infty} \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi(z_1, z_2)\,\mathrm{d}\mu^{(2,N)}_{s,\tau}(z_1, z_2) = \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi(z_1, z_2)\,\mathrm{d}\mu_s(z_1)\mu_\tau(z_2).$$

With $\varphi = \varphi_1 \otimes \varphi_2$, we get the announced result. For $s = \tau$, we obtain (4.40) for $k = 2$. According to Proposition 4.5.2, $\widehat{\mu}^N$ converges in law to μ. ■

Actually, the convergence of $(\widehat{\mu}^N)_N$ can be more precise, with an explicit rate. Let χ be a bounded Lipschitz function on $\mathbb{R}^d \times \mathbb{R}^d$. We have

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\widehat{\mu}^N_t - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu_t\right|\right] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(z_i(t) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu\right|\right] \\ &\leq \mathbb{E}[\chi(z_1(t)) - \chi(\tilde{z}_1(t))] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(\tilde{z}_i(t) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu\right|\right] \\ &\leq \|\nabla\chi\|_{L^\infty}\frac{C}{\sqrt{N}} + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(\tilde{z}_i(t) - \mathbb{E}[\chi(\tilde{z}_1(t))]\right|\right] \\ &\leq \|\nabla\chi\|_{L^\infty}\frac{C}{\sqrt{N}} + \frac{(\mathrm{Var}[\chi(\tilde{z}_1(t)])^{1/2}}{\sqrt{N}} \\ &\leq \frac{C\|\nabla\chi\|_{L^\infty} + \|\chi\|_{L^\infty}}{\sqrt{N}}, \end{aligned}$$

where we have just applied Theorem 4.5.3 to estimate the first term and the law of large numbers to the family $(\chi(\tilde{z}_i(t))_{i\in\{1,\dots,N\}}$, which Setting by construction is iid, to deal with the second term.

We now come back to the discussion of (4.34), and prove the following result

Lemma 4.5.8 *The additional term*

$$\mathcal{I}_N = \frac{\sqrt{2\gamma}}{N}\sum_{j=1}^N \int_0^t \nabla_v\varphi(z_j^N(s)) \cdot \mathrm{d}B_j(s),$$

which appear in the equation (4.34) *satisfied by the empirical measure* $\widehat{\mu}^N$ *goes to* 0 *when* N *goes to infinity in the following sense*

$$\mathbb{E}\left[|\mathcal{I}_N|\right] \leq 2\sqrt{\frac{\gamma t}{N}}\left(\|\nabla_v\varphi\|^2_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} + C^2\|\nabla_v\varphi\|^2_{\mathrm{Lip}}\right)^{1/2}.$$

Proof. We have

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{I}_N^2\right] &= \mathbb{E}\left[\left|\frac{\sqrt{2\gamma}}{N}\sum_{i=1}^N \int_0^t \nabla_v\varphi(z_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right] \\
&\leq 4\gamma\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N (\nabla_v\varphi(z_i^N(s)) - \nabla_v\varphi(\tilde{z}_i^N(s)))\,\mathrm{d}B_i(s)\right|^2\right] \\
&\qquad + 4\gamma\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right].
\end{aligned}
$$

We can get rid of the Brownian motion in the estimates of those two terms owing to

$$
\mathbb{E}\left[\left|\int_0^t X(s)\,\mathrm{d}B_s\right|^2\right] = \int_0^t \mathbb{E}\left[|X(s)|^2\right]\,\mathrm{d}s,
$$

which is a consequence of Ito's formula (see [53, Chapter 3.2.A]. On the one hand, we just have to use convexity inequality and apply Theorem 4.5.3 and we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N (\nabla_v\varphi(z_i^N(s)) - \nabla_v\varphi(\tilde{z}_i^N(s)))\,\mathrm{d}B_i(s)\right|^2\right] \\
&\qquad = \int_0^t \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \nabla_v\varphi(z_i^N(s)) - \nabla_v\varphi(\tilde{z}_i^N(s))\right|^2\right]\,\mathrm{d}s \\
&\qquad \leq \frac{1}{N}\sum_{i=1}^N \int_0^t \mathbb{E}\left[\left|\nabla_v\varphi(z_i^N(s)) - \nabla_v\varphi(\tilde{z}_i^N(s))\right|^2\right]\,\mathrm{d}s \\
&\qquad \leq \int_0^t \left(\mathbb{E}\left[\left|\nabla_v\varphi(z_1^N(s)) - \nabla_v\varphi(\tilde{z}_1^N(s))\right|\right]\right)^2\,\mathrm{d}s \\
&\qquad \leq \int_0^t \|\nabla_v\varphi\|_{\mathrm{Lip}}^2 (\mathbb{E}[|z_1(s) - \tilde{z}_1(s)| \wedge 1])^2\,\mathrm{d}s \\
&\qquad \leq \frac{C^2\|\nabla_v\varphi\|_{\mathrm{Lip}}^2 t}{N}.
\end{aligned}
$$

On the other hand, the family of the N random variables $\int_0^t \nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)$ are iid and as such it is a martingale. We thus get

$$
\mathbb{E}\left[\int_0^t \nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right] = 0.
$$

Applying the law of large numbers, we get

$$
\begin{aligned}
\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N \nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right] &= \frac{1}{N}\mathbb{E}\left[\left|\int_0^t \nabla_v\varphi(\tilde{z}_1^N(s))\,\mathrm{d}B_1(s)\right|^2\right] \\
&= \frac{1}{N}\int_0^t \mathbb{E}\left[\left|\nabla_v\varphi(\tilde{z}_1^N(s))\right|^2\right]\,\mathrm{d}s \\
&\leq \frac{t}{N}\|\nabla_v\varphi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^2.
\end{aligned}
$$

Finally, from Jensen inequality we get for all $0 \leq t \leq T$,

$$
\left(\mathbb{E}\left[|\mathcal{I}_N|\right]\right)^2 \leq \mathbb{E}\left[\mathcal{I}_N^2\right] \leq \frac{4\gamma t}{N}\left(\|\nabla_v\varphi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^2 + C^2\|\nabla_v\varphi\|_{\mathrm{Lip}}^2\right).
$$

∎

To end, we go back to (4.37), the equation satisfied by $\mu^{(1,N)}$. We use (4.55) and (4.15) with $\chi(z_1, z_2) = \nabla\Sigma(x_1 - x_2)\nabla_v\varphi(z_1)$. Hence, for all φ in $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, we obtain

$$\begin{aligned}
&\lim_{N\to\infty} \int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\nabla\sigma(x_1 - x_2).\nabla_v\varphi(z_1)\,\mathrm{d}\mu^{(2)}_{s,\tau}(z_1, z_2)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad = \int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\nabla\sigma(x_1 - x_2).\nabla_v\varphi(z_1)\,\mathrm{d}\mu_s(z_1)\,\mathrm{d}\mu_\tau(z_2)\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad = \int_0^t \int_{(\mathbb{R}^d\times\mathbb{R}^d)} \nabla_v\varphi(z).\nabla_x\mathcal{L}(\mu)(s,x)\,\mathrm{d}\mu_s(z)\,\mathrm{d}s,
\end{aligned}$$

with a rate of convergence at least $\mathscr{O}(\frac{1}{\sqrt{N}})$.

Chapitre 5

Asymptotique en temps long pour Fokker-Planck homogène avec un potentiel d'interraction régulier

Dans ce dernier article, nous nous intéressons à l'asymptotique en temps long d'une équation d'évolution portant sur la densité spatiale ρ rencontrée dans le chapitre 3. L'idée principale est d'utiliser l'entropie du système comme une fonctionnelle de Lyapunov. On montre d'abord qu'il y a bien des équilibres sous des hypothèses très générales. En restreignant notre cadre d'étude, nous montrons que la distance des solutions à l'ensemble de ces états d'équilibres tend vers 0. Nous donnons ensuite quelques critères de convergences issus de ce résultat. Afin de mieux comprendre ce que fait la solution au voisinage des états d'équilibres, nous entreprenons d'en étudier l'ensemble. Nous revisitons également quelques résultats de convergence bien connus depuis [5].

5.1 Introduction

The goal of this paper is to study the large time asymptotic property of the solution of the following equation:

$$\begin{cases} \partial_t \rho = \operatorname{div}(\nabla \rho + \rho \nabla (V + W * \rho)) & \mathbb{R}_+ \times \mathbb{R}^d \\ \rho(0) = \rho_0 & \mathbb{R}^d \end{cases} \tag{5.1}$$

where $\rho(t,x)$ denotes a density of particle at $x \in \mathbb{R}^d$ and time t. The particles are submitted to an even interaction potential W and an external potential V. (5.1) obviously preserves the mass of the system.

This equation appears as a simplified model in many fields of physics and biology. It is used to describe Poisson coupling for an electron gas. In this case the important parameters are $W = 1/|x|^{d-2}$, $d \geq 3$ and V is a confining potential, see [68] and [5]. It is also used to describe the evolution of spatially homogeneous granular media. In this case $V = 0$, $W(x) = x^3$, $d = 1$, see [9] or [50]. The potential W can also be attractive. In the Keller-Segel system the parameters are $V = 0$, $W(x) = |x|$ when $d = 1$ or $W(x) = \ln(|x|)$ when $d = 2$. In this situation, it is well known that there exists a critical mass m_c such that all solutions with initial mass $m > m_c$ concentrate in a Dirac mass in finite time, see [19]. More generally this equation can be obtained as a diffusive limit of the Vlasov-Fokker-Planck equation, see [46] or [37] for example of the Vlasov-Poisson-Fokker-Planck equation.

In many cases the trend to equilibrium has been proven in the framework of measures using the Wasserstein distance, see [21, 66, 22] . Most of these results need more or less weak convexity assertion on V or W; the hypothesis is made on their second derivative, then those results need some regularity on the potential. In order to introduce our approach, we define now the following functional that can be seen as an entropy or an energy

$$\mathscr{E}(\rho) = \int_{\mathbb{R}^d} \rho(\ln(\rho) + \frac{1}{2} W * \rho + V)\,\mathrm{d}x. \tag{5.2}$$

As we will see, its critical point are exactly the equilibrium states of (5.1) and moreover, under the following assumption

$$e^{-V} \in L^1(\mathbb{R}^d), \qquad W \geq 0, \qquad \rho_0 \geq 0, \qquad \int_{\mathbb{R}^d} \rho_0(x)\,\mathrm{d}x = m, \qquad \mathscr{E}(\rho_0) < \infty, \tag{5.3}$$

the time function $t \mapsto \mathscr{E}(\rho(t))$ is decreasing and convergent. Those simple results naturally lead us to wonder if the large time convergence of $\rho(t)$ toward an equilibrium state could be proven by using $\mathscr{E}$ as a Lyapunov functional. Although the assumption (5.3) is not far to be enough to prove the existence of equilibrium states, it seems far less possible to prove the convergence of the solution of (5.1) with only such assumption. In order to increase the information that we can get from the large time convergence of $\mathscr{E}(\rho(t))$, it seems first natural to ask the measure μ defined by

$$\mu(x) = \frac{e^{-V(x)}}{\int e^{-V(x)}\,\mathrm{d}x} dx \tag{5.4}$$

to satisfy a logarithmic Sobolev inequality. It means that we can find a constant $C > 0$ such that the following inequality holds

$$\int_{\mathbb{R}^d} |\nabla f(x)|^2\,\mathrm{d}\mu \geq C \int_{\mathbb{R}^d} f^2 \ln\Big(\frac{f^2}{\int f^2\,\mathrm{d}\mu}\Big)\,\mathrm{d}\mu$$

whenever the left hand side is finite. First studied by Gross in [49], those inequalities have since been largely used in analysis and probability (see [4] for a large review of the subject). It is well known (see [4, Cor 5.5.2]) that it is satisfied when V is such that

$$V = U + B, \qquad U \text{ is strictly convex}, \qquad B \in L^\infty(\mathbb{R}^d). \tag{5.5}$$

Requiring also W to be bounded, the best simple inequality we found to control the convergence of $\rho(t)$ to an equilibrium ρ_{eq} with the convergence of $\mathscr{E}(\rho(t))$ is the following one

$$\mathscr{E}(\rho(t)) - \mathscr{E}(\rho_{eq}) \geq \frac{1}{2}(1 + m\delta_W)\|\rho - \rho_{eq}\|^2_{L^1(\mathbb{R}^d)} \tag{5.6}$$

with

$$\delta_W := \inf_{\|h\|_{L^1} \leq 1} q_W(h), \quad q_W(h) := \int_{\mathbb{R}^d} hW * h \,\mathrm{d}x \geq -\|W\|_{L^\infty}\|h\|^2_{L^1}. \tag{5.7}$$

Under the condition $m\delta_W > -1$, we will see that it ensures the convergence to a unique equilibrium state when W is also Lipschitz. When $m\delta_W > -\frac{1}{2}$ we will see that the convergence is exponential. An analogous condition for the exponential convergence can already be found in [5]. The originality of ours is that the condition only focuses on W, while in [5] it does on W and V at the same time. Since the simpler condition to ensure the strict convexity of $\mathscr{E}$ is $m\delta_W > -1$, our condition seems closer to be the optimal one.

We then tried to know what can happen without that smallness condition. The question is harder since without such a bound on W, (5.1) has no reason to have a single equilibrium state. Still using $\mathscr{E}$ as a Lyapunov function, setting Eq, the set of all the equilibrium states and requiring W to be also Lipschitz we prove (far less directly) the weaker result

$$\lim_{t\to\infty} \inf_{\rho_{eq}\in Eq} \|\rho(t) - \rho_{eq}\|_{L^1(\mathbb{R}^d)} = 0. \tag{5.8}$$

It is (to my knowledge) the first result of this nature on this equation.

The paper is organized as follows: in the next section, we establish some general properties of (5.1) mainly concerning its equilibrium states. We see that there exists equilibrium states under very weak conditions close to (5.3), we establish their link with $\mathscr{E}$ and an operator $\mathcal{T}$ that will be very useful later. The third section is devoted to establishing the convergence results that we have just given in this introduction. The goal of the fourth one is to give some criteria that ensure that $\rho(t)$ converges under (5.8). In the last section, we tried to study the set Eq and give conditions to ensure the nature of its elements (attractive or repulsive, isolated or not).

5.2 Preliminaries

We now make the framework of our study precise and we present some simple properties of the solution of (5.1). For all mass $m \geq 0$ the following set is preserved by the evolution:

$$X := \left\{\rho \in L^1(\mathbb{R}^d) \;\middle|\; \rho \geq 0, \quad \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = m\right\}.$$

We will first assume V and W to satisfy the general assumption

$$(\mathbf{H0}) \begin{cases} V \in C(\mathbb{R}^d), \quad V(x) \xrightarrow[|x|\to\infty]{} \infty, \quad e^{-(1-\varepsilon)V} \in L^1(\mathbb{R}^d) \quad \varepsilon \in (0,1), \\ W(x) = W(-x), \qquad W \geq 0, \qquad W \in C(\mathbb{R}^d \setminus \{0\}) \cap L^1_{loc}(\mathbb{R}^d). \end{cases}$$

Reminding that $\mathscr{E}$ is defined by (5.2), for all smooth enough solution ρ of (5.1), a simple computation gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) = -\int_{\mathbb{R}^d} \frac{|\nabla\rho + \rho\nabla(V + W * \rho)|^2}{\rho}\,\mathrm{d}x. \tag{5.9}$$

Defining $\mathcal{T}$ on X by the following expression

$$\mathcal{T} : \rho \in X \mapsto \frac{m}{Z(\rho)} e^{-V-W*\rho}, \quad Z(\rho) := \int_{\mathbb{R}^d} e^{-V(x)-W*\rho(x)}\,\mathrm{d}x,$$

the time derivative (5.9) of $\mathscr{E}(\rho)$ can be recast as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) = -4\int_{\mathbb{R}^d} \left|\nabla\left(\frac{\rho}{\mathcal{T}(\rho)}\right)^{1/2}\right|^2 \mathcal{T}(\rho)\,\mathrm{d}x. \tag{5.10}$$

To clarify the connexion between the application $\mathcal{T}$, the entropy functional and the equilibrium state of (5.1), we prove the following claim:

Proposition 5.2.1 *The following assertion holds*

i) The equilibrium states of (5.1) *on X with finite entropy are the fixed points of $\mathcal{T}$.*

ii) These equilibria are also the critical points of the functional $\mathscr{E}$ on X.

iii) We remind that δ_W is defined by (5.7). *Taking $W = W_1 + W_2$, if $m\|W_1\|_{L^\infty(\mathbb{R}^d)} < 1$ and $\delta_{W_2} \geq 0$, then $\mathscr{E}$ is strictly convex on X and it contains at most one equilibrium state.*

iv) If V and W satisfies **(H0)**, *then for any mass $m > 0$ there exists at least one equilibrium state.*

Remark 5.2.2 *Taking W_1 bounded and even such that $m\|W_1\|_{L^\infty} < 1$, we choose six constants $a, b, c \geq 0$, $\gamma \in (0, d-2)$, $\sigma > 0$, $p \in \mathbb{N}$ and define W:*

$$W(x) = W_1(x) + ae^{-|x|^2/\sigma} + \frac{b}{|x|^\gamma} + c|x|^{2p}.$$

Then, applying the Fourier transform in the expression of $q_{W-W_1}(h)$, we get $\delta_{W-W_1} \geq 0$. Under **(H0)**, *there is exactly one equilibrium state of mass $m > 0$.*

In order to prove proposition 5.2.1$-$*iv*), we will prove in the appendix, the following lemma:

Lemma 5.2.3 *Compactness*
Assume **(H0)**, *then for all $r \geq 0$, the set:*

$$E_{m,r} := \{\rho \in X \mid \mathscr{E}(\rho) \leq r\}$$

is weakly compact in $L^1(\mathbb{R}^d)$.

Proof of Proposition 5.2.1. Item i) is clear.
For proving ii), we compute the first derivative of $\mathscr{E}$:

$$D\mathscr{E}(\rho)(h) = \int_{\mathbb{R}^d} (\ln(\rho) + 1 + V + W * \rho) h \, \mathrm{d}x$$

For all h such that $\int_{\mathbb{R}^d} h \, \mathrm{d}x = 0$, we can recast this expression as:

$$D\mathscr{E}(\rho)(h) = \int_{\mathbb{R}^d} (\ln(\rho) - \ln(\mathcal{T}(\rho)) h \, \mathrm{d}x.$$

Then $D\mathscr{E}(\rho) = 0$ if and only if $\ln(\rho/\mathcal{T}(\rho))$ is constant on $\mathbb{R}^d$. As ρ and $\mathcal{T}(\rho)$ have the same mass, it can only happen when $\rho = \mathcal{T}(\rho)$.
For proving iii), we next compute the second derivative of $\mathscr{E}$:

$$\begin{aligned} D^2\mathscr{E}(\rho)(h,h) &= \int_{\mathbb{R}^d} \left(\frac{h^2}{\rho} + hW * h \right) \mathrm{d}x \\ &\geq \int_{\mathbb{R}^d} \left(\frac{h^2}{\rho} + hW_1 * h \right) \mathrm{d}x \end{aligned} \tag{5.11}$$

By Young and Hölder inequalities, we get

$$\int_{\mathbb{R}^d} |hW_1 * h| \, \mathrm{d}x \leq \|W_1\|_{L^\infty(\mathbb{R}^d)} \|h\|^2_{L^1(\mathbb{R}^d)}.$$

We now control the norm of h

$$\|h\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{|h|}{\rho^{1/2}} \rho^{1/2} \, \mathrm{d}x \leq \left(\int_{\mathbb{R}^d} \frac{h^2}{\rho} \, \mathrm{d}x \right)^{1/2} \left(\int_{\mathbb{R}^d} \rho \, \mathrm{d}x \right)^{1/2} \leq m^{1/2} \|h\|_{L^2(\mathbb{R}^d; \, \mathrm{d}x/\rho)}.$$

Putting these estimates together, we get:

$$D^2\mathscr{E}(\rho)(h,h) \geq (1 - m\|W_1\|_{L^\infty(\mathbb{R}^d)}) \|h\|^2_{L^2(\mathbb{R}^d; \, \mathrm{d}x/\rho)} \geq (1-\delta) \|h\|^2_{L^2(\mathbb{R}^d; \, \mathrm{d}x/\rho)}$$

As the functional $\mathscr{E}$ is strictly convex, it cannot have more than one critical point.
We end proving iv). Owning to **(H0)**, $b = \int_{\mathbb{R}^d} e^{-V} \, \mathrm{d}x$ is well defined. We decompose the entropy as follow:

$$\mathscr{E}(\rho) = \int_{\mathbb{R}^d} \rho \ln \left(\frac{\rho}{mb^{-1}e^{-V}} \right) \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^d} \rho W * \rho \, \mathrm{d}x - m \ln(b/m)$$

As the first two terms are non-negative, we deduce that $\mathscr{E}(\rho)$ is bounded from below on X:

$$\mathscr{E}(\rho) \geq -m \ln(b/m) \tag{5.12}$$

We take a sequence $(\rho_n)_n$, such that

$$\lim_{n\to\infty} \mathscr{E}(\rho_n) = \inf_{\rho \in X} \mathscr{E}(\rho).$$

By lemma 5.2.3, we can extract a weakly convergent subsequence $(\rho_{n_k})_k$ and we set ρ^* as its limit. Also by lemma 5.2.3, the set $\{\rho \in X \mid \mathscr{E}(\rho) \leq \mathscr{E}(\rho_{n_k})\}$ is closed for all k, then

$$\mathscr{E}(\rho^*) \leq \liminf_{k\to\infty} \mathscr{E}(\rho_{n_k}) = \inf_{\rho\in X} \mathscr{E}(\rho).$$

Finally, ρ^* minimizes the entropy, then it is an equilibrium state. ∎

We now restrict the assumption we will make on the parameters and the initial data through nearly all the paper. We restrict our analysis to the L^1 initial data with finite entropy:

$$\textbf{(H1)} \qquad \rho_0 \in L^1(\mathbb{R}^d), \quad \rho_0 \geq 0, \quad \int_{\mathbb{R}^d} \rho_0(x) dx = m, \quad \mathscr{E}(\rho_0) < \infty.$$

In the proofs, we will get rid of the constant m thanks to the following property

Remark 5.2.4 *If ρ is a solution of (5.1), then for all $m > 0$, $\lambda = \frac{\rho}{m}$ satisfies:*

$$\begin{cases} \partial_t \lambda = div(\nabla \lambda + \lambda \nabla (V + (mW) * \lambda)) & \mathbb{R}_+ \times \mathbb{R}^d \\ \lambda(0) = \dfrac{\rho_0}{m} & \mathbb{R}^d \end{cases}$$

Therefore without any loss of generality, we can always assume that $\int_{\mathbb{R}^d} \rho_0(x)dx = 1$.

A interaction potential W will be supposed bounded, Lipschitz and even:

$$\textbf{(H2)} \qquad W \in W^{1,\infty}(\mathbb{R}^d), \qquad W(x) = W(-x) \quad \forall x \in \mathbb{R}^d.$$

The stronger assumption is made on the external potential V:

(H3) $V \in C(\mathbb{R}^d)$, $e^{-V} \in L^1(\mathbb{R}^d)$, the measure μ defined by (5.4) satisfies a log-Sobolev inequality of constant $C > 0$.

Let us explain how we will use that last assumption. By the Young inequality, for all ρ in X we get $\|W * \rho\|_{L^\infty(\mathbb{R}^d)} \leq m\|W\|_{L^\infty(\mathbb{R}^d)}$. A perturbative result (see [4, Thm 3.4.3]) shows that the measure $m^{-1}\mathcal{T}(\rho)(x)\,\mathrm{d}x$ satisfies also a log-Sobolev inequality with an explicit constant κ. Applying this result to $f = \left(\frac{\rho}{\mathcal{T}(\rho)}\right)^{1/2}$, we get the following lemma:

Lemma 5.2.5 *Assume* **(H3)** *and suppose that W is essentially bounded. For all $\rho \in X$, as soon as its left hand side is well defined, the following inequality holds:*

$$\int_{\mathbb{R}^d} \frac{|\nabla \rho + \rho \nabla (V + W * \rho)|^2}{\rho}\,\mathrm{d}x \geq \kappa \int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{\mathcal{T}(\rho)}\right)\mathrm{d}x,$$

where $\kappa = Ce^{-4m\|W\|_{L^\infty}}$.

The right hand side is relative to the distance between ρ and $\mathcal{T}(\rho)$. It gives another way to express the difference between those two functions by means of the Czizard-Kullback inequality, see [58] or [24]:

$$\int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{\mathcal{T}(\rho)}\right)\mathrm{d}x \geq \frac{1}{2m}\|\rho - \mathcal{T}(\rho)\|^2_{L^1(\mathbb{R}^d)}. \tag{5.13}$$

To finish this section, we define another functional, that we will work with:

$$R(\rho) := \frac{1}{2}\int_{\mathbb{R}^d} \rho W * \rho + m \ln\left(\frac{Z(\rho)}{m}\right)$$

All the results of the following section will involve some property satisfied by this functional. The important role played by R is mostly due to the following identity:

$$\begin{aligned} \mathrm{Ent}(\rho|\mathcal{T}(\rho)) &:= \int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{\mathcal{T}(\rho)}\right)\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \rho(\ln(\rho) + V + W * \rho + \ln(Z(\rho))) \\ &= \mathscr{E}(\rho) + R(\rho) \end{aligned} \tag{5.14}$$

The first term is the relative entropy of ρ with respect to $\mathcal{T}(\rho)$. Since any equilibrium is a fixed point of $\mathcal{T}$, we will see that the value of $\mathrm{Ent}(\rho|\mathcal{T}(\rho))$ gives a way to estimate the distance between ρ and the set of equilibrium states. As the expression of $\mathrm{Ent}(\rho|\mathcal{T}(\rho))$ is not to far from $\mathscr{E}$'s one, we can deduce good property on $\mathrm{Ent}(\rho|\mathcal{T}(\rho))$ from the decay and the convergence of $\mathscr{E}(\rho)$. That is why that last quantity appears naturally in our analysis and R with it. For all equilibrium state ρ_{eq}, (5.14) already ensures that $\mathscr{E}(\rho_{eq}) + R(\rho_{eq}) = 0$.

5.3 Main results

We now present our main theorem.

Theorem 5.3.1 *Large time asymptotic*
Assume **(H1)-(H3)**.

1. *For all solutions ρ to* (5.1)*, the quantity $\mathscr{E}(\rho)$ converges to a certain constant $\mathscr{E}^*$ when t goes to infinity. The set*

$$Eq(m, \mathscr{E}^*) := \left\{ \rho_{eq} \in X \;\middle|\; \rho_{eq} = \mathcal{T}(\rho_{eq}), \quad \int_{\mathbb{R}^d} \rho_{eq} = m, \quad \mathscr{E}(\rho_{eq}) = \mathscr{E}^* \right\}$$

   is non empty. Furthermore, ρ satisfies:

$$\|\rho(t) - \mathcal{T}(\rho(t))\|_{L^1(\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0 \quad \text{and} \quad \inf_{\rho_{eq} \in Eq(m,\mathscr{E}^*)} \|\rho(t) - \rho_{eq}\|_{L^1} \xrightarrow[t\to\infty]{} 0$$

2. *If $\alpha = m\delta_W > -1/2$, then there exists a unique equilibrium state ρ_{eq}. Let $\beta = \frac{1+2\alpha}{1+\alpha}$, for any solution ρ to* (5.1)*, we have the exponential decay to equilibrium:*

$$\|\rho(t) - \rho_{eq}\|_{L^1(\mathbb{R}^d)} \le \left(\frac{2m}{1+\alpha}\right)^{1/2} (\mathcal{E}(\rho_0) - \mathcal{E}^*)^{1/2} e^{-2\kappa\beta t}$$

   where κ is the constant of the log-Sobolev given by lemma 5.2.5.

Remark 5.3.2 *There are two other conditions which allow us to deduce the convergence towards an equilibrium state. Firstly, if $\alpha = m\delta_W > -1$, then proposition 5.2.1-iii) ensures that there is a unique equilibrium state. It is clear that Theorem 5.3.1$-$1) implies that $\rho(t)$ converges to this state. Secondly, the convergence can also be established if $\gamma = \frac{m}{(2\kappa)^{1/2}} \|\nabla W\|_{L^\infty(\mathbb{R}^d)} < 1$ (see remark 5.4.3). When those two conditions are satisfied, we will see in the same remark that the convergence rate is explicit:*

$$\|\rho(t) - \rho_{eq}\|_{L^1(\mathbb{R}^d)} \le \left(\frac{2m}{1+\alpha} (\mathscr{E}(\rho_0) - \mathscr{E}^*)\right)^{1/2} e^{-2\kappa(1-\gamma)t}.$$

According to remark 5.2.4, without any loss of generality, we will now assume that $m = 1$. The first item in Theorem 5.3.1 is a direct consequence of the following statement.

Lemma 5.3.3 (Time derivative spaces) *Assume* **(H1)-(H3)***, then:*

i) The function $t \mapsto \mathscr{E}(\rho(t))$ is non increasing and it converges to a certain constant $\mathscr{E}^*$ as t goes to infinity,

ii) $\partial_t \rho * W \in L^2(\mathbb{R}_+, L^\infty(\mathbb{R}^d))$,

iii) $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) \leq -4\kappa(\mathscr{E}(\rho) + R(\rho))$,

iv) $R(\rho)$ converges to $-\mathscr{E}^*$.

Proof.
Proof of i):
According to (5.9), $\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) \leq 0$, then $\mathscr{E}(\rho)$ is non increasing. Thanks to (5.12), $\mathscr{E}(\rho)$ is also bounded from below. Then $t \mapsto \mathscr{E}(\rho(t))$ converges, we set:

$$\mathscr{E}^* = \lim_{t\to\infty} \mathscr{E}(\rho(t)).$$

As $t \mapsto \mathscr{E}(\rho(t))$ is a non increasing convergent function, we deduce on its derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) \in L^1(0, +\infty). \tag{5.15}$$

Proof of ii)
Let $\chi \in W^{1,\infty}(\mathbb{R}^d)$ be a test function. According to (5.14), we have:

$$\begin{aligned}\int_{\mathbb{R}^d} \partial_t \rho \chi \,\mathrm{d}x &= \int_{\mathbb{R}^d} \mathrm{div}(\nabla\rho + \rho\nabla(V + W * \rho))\chi \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} (\nabla\rho + \rho\nabla(V + W * \rho)) \cdot \nabla\chi \,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \frac{\nabla\rho + \rho\nabla(V + W * \rho)}{\sqrt{\rho}} \cdot \sqrt{\rho}\nabla\chi \,\mathrm{d}x\end{aligned}$$

Therefore, using the Cauchy-Schwartz inequality, we have:

$$\begin{aligned}|\langle \partial_t\rho, \chi\rangle| &\leq \left(\int_{\mathbb{R}^d} \frac{|\nabla\rho + \rho\nabla(V + W * \rho)|^2}{\rho} \,\mathrm{d}x\right)^{1/2} \left(\int_{\mathbb{R}^d} \rho|\nabla\chi|^2 \,\mathrm{d}x\right)^{1/2} \\ &\leq \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho)\Big)^{1/2} \|\nabla\chi\|_{L^\infty(\mathbb{R}^d)}.\end{aligned}$$

Then, for $\chi(x) = W(y - x)$, we find:

$$\|\partial_t\rho * W\|^2_{L^\infty(\mathbb{R}^d)} \leq -\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho)\|\nabla W\|_{L^\infty(\mathbb{R}^d)} \in L^1(0, +\infty). \tag{5.16}$$

Proof of iii)
According to (5.9) we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) = -\int_{\mathbb{R}^d} \frac{|\nabla\rho + \rho\nabla(V + W * \rho)|^2}{\rho} \,\mathrm{d}x = -4\int_{\mathbb{R}^d} \left|\nabla\left(\frac{\rho}{\mathcal{T}(\rho)}\right)^{1/2}\right|^2 \mathcal{T}(\rho) \,\mathrm{d}x$$

We then apply the logarithmic Sobolev inequality for the measure $\mathcal{T}(\rho)\,\mathrm{d}x$. We get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) \leq -4\kappa\int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{\mathcal{T}(\rho)}\right) \mathrm{d}x.$$

We conclude by using (5.14).
Proof of iv)

We compute the time derivative of $R(\rho)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R(\rho) &= \int \left(\rho W * \partial_t \rho + \frac{\partial_t Z(\rho)}{Z(\rho)}\right)\,\mathrm{d}x \\ &= \int \rho W * \partial_t \rho \,\mathrm{d}x - \frac{1}{Z(\rho)}\int_{\mathbb{R}^d} e^{-V-W*\rho} W * \partial_t \rho \,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (\rho - \mathcal{T}(\rho)) W * \partial_t \rho \,\mathrm{d}x.\end{aligned}$$

Then we have:

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}R(\rho)\Big| \leq \|\rho - \mathcal{T}(\rho)\|_{L^1(\mathbb{R}^d)} \|\partial_t \rho * W\|_{L^\infty(\mathbb{R}^d)}$$

Using (5.13) and (5.14), we control the first term in the product:

$$\|\rho - \mathcal{T}(\rho)\|_{L^1(\mathbb{R}^d)} \leq \left(2\int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{\mathcal{T}(\rho)}\right)\,\mathrm{d}x\right)^{1/2} = \sqrt{2}(\mathscr{E} + R)^{1/2} \leq \left(-\frac{1}{2\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho)\right)^{1/2}$$

Finally, using (5.16), we find for $\frac{\mathrm{d}}{\mathrm{d}t}R(\rho)$ the following estimate:

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}R(\rho)\Big| \leq -\frac{1}{(2\kappa)^{1/2}}\|\nabla W\|_{L^\infty(\mathbb{R}^d)}\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(\rho) \in L^1(0,+\infty)$$

Consequently, $R(\rho(t))$ converges when t goes to infinity. As $\mathscr{E}(\rho(t))$ goes to $\mathscr{E}^*$ and $\mathscr{E}(\rho) + R(\rho)$ belongs to $L^1(\mathbb{R}_+)$, $R(\rho(t))$ goes to $-\mathscr{E}^*$ when t goes to infinity. More precisely, we have:

$$|R(\rho) + \mathscr{E}^*| \leq \frac{1}{(2\kappa)^{1/2}}\|\nabla W\|_{L^\infty(\mathbb{R}^d)}(\mathscr{E}(\rho) - \mathscr{E}^*) \tag{5.17}$$

■

Proof of Theorem 5.3.1.

First statement:

The first part of the theorem is almost already proven: according with Lemma 5.3.3, $\mathscr{E}(\rho) + R(\rho)$ goes to 0 when t goes to infinity. Using (5.14) and the Czizard-Kullback inequality, we find:

$$\|\rho(t) - \mathcal{T}(\rho(t))\|_{L^1(\mathbb{R}^d)} \leq \sqrt{2}(\mathscr{E}(\rho) + R(\rho))^{1/2} \xrightarrow[t\to\infty]{} 0 \tag{5.18}$$

As the equilibrium are precisely the fixed points of $\mathcal{T}$, using the compactness property, we prove that ρ tends to be closer and closer to the set of equilibrium states.

Compactness:

We show the relative compactness of the set

$$K = \left\{\mathcal{T}(\rho) \quad \Big| \quad \|\rho\|_{L^1(\mathbb{R}^d)} \leq 1\right\}. \tag{5.19}$$

For all sequences $(\mathcal{T}(\rho_n))_n$ of elements of K, the sequence $(W * \rho_n)_n$ is uniformly bounded in $W^{1,\infty}(\mathbb{R}^d)$. Then, using the Arzela-Ascoli theorem, it is compact in $C(B(0,R))$ for all $0 < R < \infty$. A diagonal argument allows us to extract a subsequence $(n_k)_k$ such that $W * \rho_{n_k}$ converges in $L^\infty(B(0,R))$ for all $R > 0$. For all k we have:

$$e^{-V-\|W\|_{L^\infty(\mathbb{R}^d)}} \leq e^{-V-W*\rho_{n_k}} \leq e^{-V+\|W\|_{L^\infty(\mathbb{R}^d)}}$$

As e^{-V} belongs to $L^1(\mathbb{R}^d)$, according to the dominated convergence theorem, the sequence $e^{-V-W*\rho_{n_k}}$ converges in $L^1(\mathbb{R}^d)$. Then, $\mathcal{T}(\rho_{n_k}) = \left(\int_{\mathbb{R}^d} e^{-V-W*\rho_{n_k}}\,\mathrm{d}x\right)^{-1} e^{-V-W*\rho_{n_k}}$ also converges in $L^1(\mathbb{R}^d)$.

Uniform continuity:

Let B be the unit ball of $L^1(\mathbb{R}^d)$, we establish Lipschitz estimates on R and $\mathcal{T}$ on B. For all ρ_1, ρ_2 in B, we have:

$$\frac{\mathcal{T}(\rho_1)}{\mathcal{T}(\rho_2)} = e^{-W*(\rho_1-\rho_2)} \int_{\mathbb{R}^d} e^{W*(\rho_1-\rho_2)} \mathcal{T}(\rho_1)\,\mathrm{d}x \tag{5.20}$$

Then,

$$\begin{aligned} |\mathcal{T}(\rho_1) - \mathcal{T}(\rho_2)| &\le \mathcal{T}(\rho_2) \|1 - e^{-W*(\rho_1-\rho_2)} \int_{\mathbb{R}^d} e^{W*(\rho_1-\rho_2)} \mathcal{T}(\rho_1)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}x \\ &\le \mathcal{T}(\rho_2)\left(\|1 - e^{-W*(\rho_1-\rho_2)}\|_{L^\infty(\mathbb{R}^d)} + \|1 - e^{W*(\rho_1-\rho_2)}\|_{L^\infty(\mathbb{R}^d)} e^{2\|W\|_{L^\infty(\mathbb{R}^d)}}\right) \\ &\le \mathcal{T}(\rho_2)(1 + e^{2\|W\|_{L^\infty(\mathbb{R}^d)}}) e^{2\|W\|_{L^\infty(\mathbb{R}^d)}} \|W\|_{L^\infty(\mathbb{R}^d)} \|\rho_1 - \rho_2\|_{L^1(\mathbb{R}^d)} \end{aligned} \tag{5.21}$$

which ensures that $\mathcal{T}$ is uniformly continuous on B. For R , we have:

$$R(\rho_1) - R(\rho_2) = \ln\left(\int_{\mathbb{R}^d} e^{W*(\rho_2-\rho_1)} \mathcal{T}(\rho_1)\,\mathrm{d}x\right) + \frac{1}{2}\int_{\mathbb{R}^d} (\rho_1 + \rho_2) W * (\rho_1 - \rho_2)\,\mathrm{d}x$$

We deduce that:

$$|R(\rho_1) - R(\rho_2)| \le 2\|W\|_{L^\infty(\mathbb{R}^d)} \|\rho_1 - \rho_2\|_{L^1(\mathbb{R}^d)}$$

As R and $\mathcal{T}$ are uniformly continuous on B, (5.18) and $iv)$ in lemma 5.3.3 gives:

$$\|\mathcal{T} \circ \mathcal{T}(\rho(t)) - \mathcal{T}(\rho(t))\|_{L^1(\mathbb{R}^d)} \xrightarrow[t\to\infty]{} 0 \quad , \quad |R(\mathcal{T}(\rho)) + \mathscr{E}^*| \xrightarrow[t\to\infty]{} 0 \tag{5.22}$$

Proof of 1)*:*

Let us introduce the non negative continuous functional φ on $L^1(\mathbb{R}^d)$ defined by:

$$\varphi(\rho) = \|\rho - \mathcal{T}(\rho)\|_{L^1(\mathbb{R}^d)} + |R(\rho) + \mathscr{E}^*|. \tag{5.23}$$

According to (5.22) $\varphi(\mathcal{T}(\rho))$ converges to 0 when t goes to infinity. For any $t \ge 0$, $\mathcal{T}(\rho(t))$ belongs to the compact set $\overline{K}$ of $L^1(\mathbb{R}^d)$, we deduce

$$\inf_{h\in\overline{K}} \varphi(h) = 0.$$

As φ is continuous on $\overline{K}$, we deduce that $\varphi^{-1}(\{0\})$ is non empty. As any equilibrium of (5.1) is a fixed point of $\mathcal{T}$, (5.14) implies that all equilibrium states $\rho_{eq} \in Eq(1, \mathscr{E}^*)$ satisfy $R(\rho_{eq}) + \mathscr{E}(\rho_{eq}) = 0$. Then we have exactly $\varphi^{-1}(\{0\}) = Eq(1, \mathscr{E}^*)$.

Moreover, since φ is continuous on the compact set $\overline{K}$, for all $\epsilon > 0$, we can find $\delta > 0$ such that for all h in $\overline{K}$:

$$\text{if} \inf_{\rho_{eq}\in Eq(m,\mathscr{E}^*)} \|h - \rho_{eq}\|_{L^1} \ge \epsilon \quad \text{then} \quad \varphi(h) \ge \delta \tag{5.24}$$

(δ is the minimum of φ over the compact set $\{h \in \overline{K} \mid \inf_{\rho_{eq}\in Eq(m,\mathscr{E}^*)} \|h - \rho_{eq}\|_{L^1} \ge \epsilon\}$). Since $\varphi(\mathcal{T}(\rho))$ converges to 0 when t goes to infinity, the contrapositive of (5.24) allows us to deduce

$$\inf_{\rho_{eq}\in Eq(m,\mathscr{E}^*)} \|\mathcal{T}(\rho(t)) - \rho_{eq}\|_{L^1} \xrightarrow[t\to\infty]{} 0.$$

By (5.18), we conclude:

$$\inf_{\rho_{eq}\in Eq(m,\mathscr{E}^*)} \|\rho(t)-\rho_{eq}\|_{L^1} \xrightarrow[t\to\infty]{} 0.$$

Proof of 2)
We ensure that under the smallness condition on W, $R-\mathscr{E}^*$ is sufficiently small compared with $\mathscr{E}-\mathscr{E}^*$ and it will be possible to apply Grönwall's lemma. For $\mathscr{E}$, since $\mathscr{E}^*+R(\rho_{eq})=0$, we find

$$\begin{aligned}
\mathscr{E}(\rho)-\mathscr{E}^* &= \int_{\mathbb{R}^d} \rho(\ln(\rho)-\frac{1}{2}W*\rho+V)\,\mathrm{d}x + \ln(Z(\rho_{eq})) + \frac{1}{2}\int_{\mathbb{R}^d}\rho_{eq}W*\rho_{eq}\,\mathrm{d}x \qquad (5.25)\\
&= \int_{\mathbb{R}^d}\rho\ln\left(\frac{\rho}{\rho_{eq}}\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}(\rho-\rho_{eq})W*(\rho-\rho_{eq})\,\mathrm{d}x\\
&\geq \frac{1}{2}(1+\delta_W)\|\rho-\rho_{eq}\|_{L^1}^2.
\end{aligned}$$

Similarly, we find for R

$$\begin{aligned}
\mathscr{E}^*+R(\rho) &= \int_{\mathbb{R}^d}\rho_{eq}(\ln(\rho_{eq})-\frac{1}{2}W*\rho_{eq}+V)\,\mathrm{d}x + \ln(Z(\rho)) + \frac{1}{2}\int_{\mathbb{R}^d}\rho W*\rho\,\mathrm{d}x \qquad (5.26)\\
&= \int_{\mathbb{R}^d}\rho_{eq}\ln\left(\frac{\rho_{eq}}{\mathcal{T}(\rho)}\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}(\rho-\rho_{eq})W*(\rho-\rho_{eq})\,\mathrm{d}x\\
&\geq \frac{1}{2}\delta_W\|\rho-\rho_{eq}\|_{L^1}^2\\
&\geq \frac{\delta_W}{1+\delta_W}(\mathscr{E}(\rho)-\mathscr{E}^*),
\end{aligned}$$

where we have used $\int_{\mathbb{R}^d}\rho_{eq}\ln\left(\frac{\rho_{eq}}{\mathcal{T}(\rho)}\right)\mathrm{d}x \geq 0$ and applied (5.25). Going back to Lemma 5.3.3$-iii$), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{E}(\rho)-\mathscr{E}^*) \leq -4\kappa\left(1+\frac{\delta_W}{1+\delta_W}\right)(\mathscr{E}(\rho)-\mathscr{E}^*).$$

Let $\beta=\frac{1+2\delta_W}{1+\delta_W}$, Assuming $\delta_W>-1/2$, β is positive and the Grönwall lemma leads to:

$$\mathscr{E}(\rho(t))-\mathscr{E}^* \leq (\mathscr{E}(\rho(0))-\mathscr{E}^*)e^{-4\kappa\beta t}$$

Finally with (5.25), we get

$$\|\rho-\rho_{eq}\|_{L^1} \leq \left(\frac{2}{1+\delta_W}(\mathscr{E}(\rho_0)-\mathscr{E}(\rho_{eq}))\right)^{1/2} e^{-2\kappa\beta t}.$$

∎

Remark 5.3.4 *The condition $m\delta_W>-1/2$ in Theorem 5.3.1 is quite similar to the condition needed to prove that the entropy is strictly convex in Proposition 5.2.1. One could think that the difference is only due to the simple bounds established in* (5.26)*. A more precise study shows that we can not do better. We define the functional*

$$J(\rho)=\int_{\mathbb{R}^d}\rho_{eq}\ln\left(\frac{\rho_{eq}}{\mathcal{T}(\rho)}\right)\mathrm{d}x.$$

Using the fact that $\rho_{eq} = \mathcal{T}(\rho_{eq})$ *and* (5.20), *it can be recast as*

$$J(\rho) = -\int_{\mathbb{R}^d} \rho_{eq} W * (\rho_{eq} - \rho)\,\mathrm{d}x + \ln\left(\int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})}\,\mathrm{d}x\right).$$

We compute the first two differentials of J *acting on* $L^1(\mathbb{R}^d)$. *On the one hand we have*

$$\begin{aligned} DJ(\rho).h &= \int_{\mathbb{R}^d} \rho_{eq} W * h\,\mathrm{d}x - \left(\int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})}\,\mathrm{d}x\right)^{-1} \int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})} W * h\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (\rho_{eq} - \mathcal{T}(\rho)) W * h\,\mathrm{d}x. \end{aligned}$$

For $\rho = \rho_{eq}$, *we already notice that* $DJ(\rho) = 0$. *On the other hand, we have*

$$\begin{aligned} D^2 J(\rho).(h,h) &= \left(\int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})}\,\mathrm{d}x\right)^{-1} \int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})} (W*h)^2\,\mathrm{d}x \\ &\quad - \left(\int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})}\,\mathrm{d}x\right)^{-2} \left(\int_{\mathbb{R}^d} \rho_{eq} e^{-W*(\rho-\rho_{eq})} W * h\,\mathrm{d}x\right)^2 \\ &= \int_{\mathbb{R}^d} (W*h)^2 \mathcal{T}(\rho)\,\mathrm{d}x - \left(\int_{\mathbb{R}^d} (W*h)\mathcal{T}(\rho)\,\mathrm{d}x\right)^2 \\ &= \mathrm{Var}_{\mathcal{T}(\rho)}[W*h]. \end{aligned}$$

Then, $J(\rho)$ *can also be written:*

$$J(\rho) = \int_0^1 \int_0^s \mathrm{Var}_{\mathcal{T}((1-\tau)\rho_{eq}+\tau\rho)}[W * (\rho - \rho_{eq})]\,\mathrm{d}\tau\,\mathrm{d}s. \tag{5.27}$$

Using (5.20), *we establish the following inequality:*

$$e^{-2\tau\|W*(\rho-\rho_{eq})\|_{L^\infty(\mathbb{R}^d)}} \mathrm{Var}_{\rho_{eq}} \le \mathrm{Var}_{\mathcal{T}((1-\tau)\rho_{eq}+\tau\rho)} \le e^{2\tau\|W*(\rho-\rho_{eq})\|_{L^\infty(\mathbb{R}^d)}} \mathrm{Var}_{\rho_{eq}}.$$

Going back to (5.27) *and using Theorem 5.3.1*-1), *we deduce that*

$$J(\rho(t)) \underset{t\to\infty}{\sim} \frac{1}{2} \mathrm{Var}_{\rho_{eq}}[W * (\rho(t) - \rho_{eq})].$$

For all h *in* $L^1(\mathbb{R}^d)$ *such that* $\int h = 0$, *we know that*

$$\textit{if} \quad \mathrm{Var}_{\rho_{eq}}[W*h] = 0 \quad \textit{then} \quad \int_{\mathbb{R}^d} hW * h\,\mathrm{d}x = 0.$$

Nevertheless, this is not enough to control $\int_{\mathbb{R}^d} hW * h\,\mathrm{d}x$ *by* $\mathrm{Var}_{\rho_{eq}}[W*h]$.

5.4 Criteria of convergence

The conclusion of Theorem 5.3.1-1) does not establish whether or not $\rho(t)$ converges when t goes to infinity. We give now some conditions which complement this result and establish the convergence.

Global Criteria

Theorem 5.4.1 *(Criteria of convergence)*
Assume **(H1)-(H3)**. *If one of these following three conditions is satisfied*

1. $Eq(m, \mathscr{E}^*)$ *is totally disconnected in* $L^1(\mathbb{R}^d)$,

2. $\rho(t)$ *converges weakly in* $\mathcal{D}'(\mathbb{R}^d)$,

3. $\|\rho - \mathcal{T}(\rho)\|_{L^p(\mathbb{R}^d)}$ *belongs to* $L^1(\mathbb{R}_+)$ *for some* p *in* $[1, +\infty]$,

then we can find ρ_{eq} *in* $Eq(m, \mathscr{E}^*)$ *such that* $\rho(t)$ *goes to* ρ_{eq} *in* $L^1(\mathbb{R}^d)$ *when* t *goes to infinity.*

Remark 5.4.2 *Unlike Theorem 5.3.1*–2)*, this theorem makes us able to prove the strong convergence of* ρ *even if the equilibrium state is not unique.*

Remark 5.4.3 *Theorem 5.4.1*–3) *gives another condition on the parameters to ensure the convergence of* $\rho(t)$*. Indeed, assume that* $\gamma = \frac{m}{(2\kappa)^{1/2}} \|\nabla W\|_{L^\infty(\mathbb{R}^d)} < 1$*, then, from* (5.17) *and lemma 5.3.3*–3)*, we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{E}(\rho) - \mathscr{E}^*) \leq -4\kappa(1-\gamma)(\mathscr{E}(\rho) - \mathscr{E}^*)$$

The Grönwall lemma ensures that

$$\mathscr{E}(\rho(t)) - \mathscr{E}^* \leq (\mathscr{E}(\rho_0) - \mathscr{E}^*)e^{-4\kappa(1-\gamma)t}. \tag{5.28}$$

Then, we also have

$$\mathscr{E}(\rho(t)) - R(\rho(t)) \leq (1+\gamma)(\mathscr{E}(\rho_0) - \mathscr{E}^*)e^{-4\kappa(1-\gamma)t}.$$

With Czizard-Kullback inequality, we get

$$\|\rho(t) - \mathcal{T}(\rho(t))\|_{L^1(\mathbb{R}^d)} \leq (2m(1+\gamma)(\mathscr{E}(\rho_0) - \mathscr{E}^*))^{1/2} \, e^{-2\kappa(1-\gamma)t}.$$

Finally, $\rho - \mathcal{T}(\rho)$ *belongs to* $L^1(\mathbb{R}_+, L^1(\mathbb{R}^d))$ *and we apply Theorem 5.4.3*–3)*.*
To complement this remark, we just point out that under the additional condition $\alpha = m\delta_W > -1$*, the convergence to the unique equilibrium state of the system is explicit. Indeed, using* (5.25)*, we deduce from* (5.28)*, the following estimate:*

$$\|\rho(t) - \rho_{eq}\|_{L^1(\mathbb{R}^d)} \leq \left(\frac{2m}{1-\alpha}(\mathscr{E}(\rho_0) - \mathscr{E}^*)\right)^{1/2} e^{-2\kappa(1-\gamma)t}.$$

In order to prove Theorem 5.4.1-3), we will need the following lemma.

Lemma 5.4.4 *Let* X *be a separated topological space and let* Y *be a compact set of* X*. Let* γ *be a continuous function from* $\mathbb{R}_+$ *to* X*. We suppose that for all open neighbourhoods* $\mathcal{U}$ *of* Y*, we can find* $\eta > 0$ *such that* $\{\gamma(t) \mid t \geq \eta\}$ *is contained in* $\mathcal{U}$*. Then the set of limit points*

$$Adh(\gamma) := \bigcap_{n \geq 0} \overline{\{\gamma(t) \mid t \geq n\}}$$

is a non empty connected subset of Y*. All its open neighbourhoods* $\mathcal{U}$ *also contain a set* $\{\gamma(t) \mid t \geq \eta\}$ *for some* $\eta > 0$*.*

Proof of theorem 5.4.1.
Proof of 1)

We first check that the hypothesis of Lemma 5.4.4 are satisfied. If φ, K are respectively defined by (5.23),(5.19), then we have

$$Eq(m, \mathscr{E}^*) = \left\{ \rho \in \overline{K} \quad | \quad \varphi(\rho) = 0 \right\}.$$

As $\overline{K}$ is a compact set of $L^1(\mathbb{R}^d)$ and φ a continuous function on $L^1(\mathbb{R}^d)$, it ensures that $Eq(m, \mathscr{E}^*)$ is also a compact set of $L^1(\mathbb{R}^d)$. According to Theorem 5.3.1, the distance between $\rho(t)$ and $Eq(m, \mathscr{E}^*)$ goes to 0 when t goes to infinity. Then, according to Lemma 5.4.4, the distance between $\rho(t)$ and a connected component of $Eq(m, \mathscr{E}^*)$ goes to 0. As $Eq(m, \mathscr{E}^*)$ is totally disconnected, we can find ρ_{eq} in $Eq(m, \mathscr{E}^*)$ such that $\rho(t)$ goes to ρ_{eq} when t goes to infinity.

Proof of 2)

Suppose that $\rho(t)$ converges to some distribution ρ^* in $\mathcal{D}'(\mathbb{R}^d)$. Picking χ in $C_0(\mathbb{R}^d)$ (the space of continuous function which goes to 0 at infinity) and $n > 0$, we can find χ_n in $\mathcal{D}(\mathbb{R}^d)$ such that $\|\chi - \chi_n\|_{L^\infty(\mathbb{R}^d)} < 1/n$. As $\|\rho(t)\|_{L^1(\mathbb{R}^d)} = \|\rho_0\|_{L^1(\mathbb{R}^d)}$, for all $t \geq 0$, we get $| < \rho(t), \chi > - < \rho(t), \chi_n > | \leq \|\rho_0\|_{L^1(\mathbb{R}^d)}/n$. Since the time function $t \mapsto < \rho(t), \chi >$ is a uniform limit of convergent function, it also converges when t goes to infinity. we deduce that $\rho(t)$ converges weakly in $C_0(\mathbb{R}^d)'$. As $\mathcal{T}(\rho(t))$ belongs to $\overline{K}$ which is a compact set of $L^1(\mathbb{R}^d)$ (see proof of Theorem 5.3.1$-$1), we can find a subsequence $(t_n)_{n\geq 0}$ such that $\mathcal{T}(\rho(t_n))$ converges in $L^1(\mathbb{R}^d)$. By (5.18), it also converges to ρ^*. Then we have

$$\int_{\mathbb{R}^d} \rho^*(x)\,\mathrm{d}x = \lim_{t\to\infty} \int_{\mathbb{R}^d} \mathcal{T}(\rho(t_n))(x)\,\mathrm{d}x = m.$$

This allows us to deduce that $\rho(t)$ converges weakly in $C_b(\mathbb{R}^d)'$ where $C_b(\mathbb{R}^d)$ is the space of continuous bounded functions on $\mathbb{R}^d$ (see [65, Thm II.6.8]). Owning to **(H1)**, $W \in C_b(\mathbb{R}^d)$ then for all x we have:

$$\lim_{t\to\infty} W * \rho(t,x) = W * \rho^*(x)$$

According with the dominated convergence theorem $e^{-V+W*\rho(t)}$ converges to $e^{-V+W*\rho^*}$, then $\mathcal{T}(\rho(t))$ goes to $\mathcal{T}(\rho^*)$ in $L^1(\mathbb{R}^d)$. Finally, by (5.18), we have

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} \mathcal{T}(\rho(t)) = \rho^* \quad \text{in} \quad L^1(\mathbb{R}^d).$$

As $\mathcal{T}$ is continuous on $L^1(\mathbb{R}^d)$, ρ^* is an equilibrium state. As R is continuous on $L^1(\mathbb{R}^d)$, because of Lemma 5.3.3$-$4), $R(\rho^*) = -\mathscr{E}^*$, then ρ^* belongs to $Eq(m, \mathscr{E}^*)$.

Proof of 3)

Take χ in $\mathcal{D}(\mathbb{R}^d)$. We make the following computation:

$$\begin{aligned} < \partial_t\rho, \chi > &= -\int_{\mathbb{R}^d} (\nabla\rho + \rho\nabla(V + W * \rho)).\nabla\chi\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} (\nabla(\rho - \mathcal{T}(\rho)) + (\rho - \mathcal{T}(\rho))\nabla(V + W * \rho)).\nabla\chi\,\mathrm{d}x + 0 \\ &= \int_{\mathbb{R}^d} (\rho - \mathcal{T}(\rho))(\Delta\chi - \nabla(V + W * \rho).\nabla\chi)\,\mathrm{d}x. \end{aligned}$$

Then, the following estimate holds:

$$| < \partial_t\rho, \chi > | \leq \|\rho - \mathcal{T}(\rho)\|_{L^p(\mathbb{R}^d)}(\|\Delta\chi - \nabla V.\nabla\chi\|_{L^{p'}(\mathbb{R}^d)} + m\|\nabla W\|_{L^\infty}\|\nabla\chi\|_{L^{p'}(\mathbb{R}^d)})$$

Finally, $\frac{\mathrm{d}}{\mathrm{d}t} < \rho, \chi >$ belongs to $L^1(\mathbb{R}_+)$. It proves that $\rho(t)$ converges weakly in $\mathcal{D}'(\mathbb{R}^d)$. By 2) we conclude that 3) is also true. ∎

Proof of Lemma 5.4.4.

We first prove the compactness of L defined by:

$$L := Y \cup \{\gamma(t) \mid t \geq 0\}.$$

Take $\bigcup_{\alpha\in A} U_\alpha$, an open covering of L. It is an open covering of Y and so we can find $\alpha_1, .., \alpha_n$ such that $\bigcup_{i=1}^n U_{\alpha_i}$ contains Y. As the set $\bigcup_{i=1}^n U_{\alpha_i}$ is an open neighbourhood of Y, we can find $\eta > 0$ such that it also contains $\{\gamma(t) \mid t \geq \eta\}$. Since γ is continuous, the set $\{\gamma(t) \mid 0 \leq t \leq \eta\}$ is also a compact set covered by $\bigcup_{\alpha\in A} U_\alpha$, then we can also find $\alpha_{n+1}, .., \alpha_m$ such that it is contained by $\bigcup_{i=n+1}^m U_{\alpha_i}$. Finally, we have:

$$L \subset \bigcup_{i=1}^{m} U_{\alpha_i}.$$

We can already deduce that $Adh(\gamma)$ is non empty (and compact) as any non increasing intersection of sub compact sets of L.

Take $x \notin Y$. As X is separated, we can find two open sets $\mathcal{U}$ and $\mathcal{O}$ such that $Y \subset \mathcal{U}$, $x \in \mathcal{O}$ and $\mathcal{U} \cap \mathcal{O} = \emptyset$. We then can find $\eta > 0$ such that $\mathcal{U}$ contains $\{\gamma(t) \mid t \geq \eta\}$. For such η, we have $\{\gamma(t) \mid t \geq \eta\} \cap \mathcal{O} = \emptyset$, then $x \notin Adh(\gamma)$. It ensures that $Adh(\gamma) \subset Y$.

We now suppose that we can find $\mathcal{O}_1, \mathcal{O}_2$, two non empty disjoint open subsets of $Adh(\gamma)$ such that $Adh(\gamma) = \mathcal{O}_1 \cup \mathcal{O}_2$. The two sets $\mathcal{O}_1$ and $\mathcal{O}_2$ have an open complement in $Adh(\gamma)$, then they are both closed and then compact. As X is separated, we can find $\mathcal{U}_1$ and $\mathcal{U}_2$, two disjoint open sets of X such that $\mathcal{O}_i \subset \mathcal{U}_i$ for $i = 1, 2$. As $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint and open, $\overline{\mathcal{U}_1} \cap \mathcal{U}_2 = \emptyset$. Take now $(y_1, y_2) \in \mathcal{O}_1 \times \mathcal{O}_2$. The definition of $Adh(\gamma)$ allows us to construct an increasing sequence $(t_n)_{n\geq 0}$ going to infinity such that

$$t_{2n+1} \in \mathcal{U}_1, \quad t_{2n} \in \mathcal{U}_2 \quad n \in \mathbb{N}.$$

For all $n \geq 0$ we split $[t_{2n}, t_{2n+1}]$ in three disjoint sets:

$$[t_{2n}, t_{2n+1}] = \{s \mid \gamma(s) \in \mathcal{U}_1\} \cup \{s \mid \gamma(s) \in \complement\overline{\mathcal{U}_1}\} \cup \{s \mid \gamma(s) \in \partial\mathcal{U}_1\}.$$

The first two ones are open and non empty by choice of t_{2n} and t_{2n+1}. As $[t_{2n}, t_{2n+1}]$ is connected, the third one is also non empty. It allows us to construct a sequence $(s_n)_{n\geq 0}$ such that $t_{2n} < s_n < t_{2n+1}$ and $\gamma(s_n) \in \partial\mathcal{U}_1$ for all n. As s_n goes to infinity and $(s_n)_{n\geq 0} \in (L \cap \partial\mathcal{U}_1)^{\mathbb{N}}$, we have

$$\emptyset \neq \bigcap_{n\geq 0} \overline{\{\gamma(s_k) \mid k \geq n\}} \subset \bigcap_{n\geq 0} \overline{\{\gamma(t) \mid t \geq n\}} = Adh(\gamma).$$

It proves that $Adh(\gamma) \cap \partial\mathcal{U}_1 \neq \emptyset$. Take x in $Adh(\gamma) \cap \partial\mathcal{U}_1$. It cannot be in $\mathcal{U}_1$ because it is open, it can neither be in $\mathcal{U}_2$ because $\overline{\mathcal{U}_1} \cap \mathcal{U}_2 = \emptyset$, therefore we cannot have $Adh(\gamma) \subset \mathcal{U}_1 \cup \mathcal{U}_2$. As we have supposed $Adh(\gamma) = \mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{O}_i \subset \mathcal{U}_i$ for $i = 1, 2$; we have a contradiction. We conclude that $Adh(\gamma)$ is connected.

Take an open set $\mathcal{U}$ such that for any $\eta > 0$, $\{\gamma(t) \mid t \geq \eta\} \not\subset \mathcal{U}$, then we can find a sequence $(t_n)_n$ going to infinity such that $\gamma(t_n) \notin \mathcal{U}$. As a sequence of points of the compact set $L \cap \complement\mathcal{U}$, it has an accumulation point. This point allows us to deduce that $Adh(\gamma) \cap \complement\mathcal{U} \neq \emptyset$. Finally, there is no such open set containing $Adh(\gamma)$. We conclude that all open neighbourhoods of $Adh(\gamma)$ contain the whole set $\{\gamma(t) \mid t \geq \eta\}$ for some $\eta > 0$. ■

Local criteria

When there is more than one equilibrium state, some of them can be unstable while some others can be stable. It would be interesting to know how to determine each case. Besides, the first point of the last theorem, involves knowing whether or not an equilibrium is isolated in $Eq(m, \mathscr{E}^*)$. In order to give a (partial) answer to these questions, choosing an equilibrium state ρ_{eq}, we restrict the study on the square integrable function for the measure $\mathrm{d}x/\rho_{eq}$. Our conditions will concern the spectrum of the following operator:

$$\mathcal{L}(h) = \rho_{eq} W * h - \rho_{eq} \int_{\mathbb{R}^d} W * h \rho_{eq} \,\mathrm{d}x.$$

One can check that it is self-adjoint on the Hilbert space

$$\mathcal{H} = \left\{ h \in L^2(\mathbb{R}^d; \,\mathrm{d}x/\rho_{eq}) \,\middle|\, \int_{\mathbb{R}^d} h = 0 \right\}$$

and thanks to **(H2)**, it is also compact. The reasons that make this operator interesting are the following.

Proposition 5.4.5 *Assume* **(H2)-(H3)***, the following assertions hold:*

i) $\mathcal{L}(h)$ *is compact and self adjoin on* $\mathcal{H}$*, its spectrum* $\sigma(\mathcal{L})$ *is a sequence of eigenvalues* λ_n *decreasing to* 0 *in absolute value.*

ii) $D\mathcal{T}(\rho_{eq})(h) = -\mathcal{L}(h)$

iii) $D^2\mathscr{E}(\rho_{eq})(h,h) = \langle h, (Id + \mathcal{L})(h) \rangle_{\mathcal{H}}$

iv) The linearised equation of (5.1) *around* ρ_{eq} *is* $\partial_t h = div\left(\nabla\left(\dfrac{(\mathcal{L}+Id)(h)}{\rho_{eq}}\right)\rho_{eq}\right)$

The following theorem is a direct consequence of the last proposition:

Theorem 5.4.6 *Assume* **(H2)-(H3)***, the nature of* ρ_{eq} *is characterised by* $\mathcal{L}$ *as follow:*

1. *If* $\min(\sigma(\mathcal{L})) > -1$*, then* ρ_{eq} *is attractive.*
2. *If* $\min(\sigma(\mathcal{L})) < -1$*, then* ρ_{eq} *is unstable.*
3. *If* $-1 \notin \sigma(\mathcal{L})$ *then* ρ_{eq} *is an isolated equilibrium state of* (5.1) *in* $L^1(\mathbb{R}^d)$*.*

We first prove the theorem 5.4.6:
Proof. *Proof of* $1) - 2)$*:*
Since by proposition 5.2.1-*i*), the equilibrium states are the critical points of $\mathscr{E}$, if we have $\min(\sigma(\mathcal{L})) > -1$, then by proposition 5.4.5-*iii*), $D^2\mathscr{E}(\rho_{eq})$ is a strictly positive quadratic form. We deduce that ρ_{eq} is a strict local minimum of $\mathscr{E}$. As $\mathscr{E}(\rho(t))$ decreases in time for all solutions ρ of (5.1), ρ_{eq} is attractive.

On the other way, if $\min(\sigma(\mathcal{L})) < -1$, then we can find an eigenvalue $\omega < -1$ and an eigenvector h_ω such that $\mathcal{L}(h_\omega) = \omega h_\omega$. By proposition 5.2.1-$i$) and by proposition 5.4.5-iii), setting $g(s) = \mathscr{E}(\rho_{eq} + s h_\omega)$, we have

$$g'(0) = 0, \qquad g''(0) = \langle h_\omega, (Id + \mathcal{L})(h_\omega)\rangle_{\mathcal{H}} = (\omega + 1)\|h_\omega\|^2_{L^2(\mathbb{R}^d;\,\mathrm{d}x/\rho_{eq})} < 0.$$

It means that g has a local maximum in 0. As $\mathscr{E}(\rho(t))$ decreases in time for all solution ρ of (5.1), ρ_{eq} is unstable.

Proof of 3)*:*

We define F, acting on $\mathcal{H}$ as

$$F(h) = \mathcal{T}(\rho_{eq} + h) - \rho_{eq} - h.$$

As ρ_{eq} is an equilibrium state, it is clear that we have

$$F(0) = 0, \qquad DF(0) = -\mathcal{L} - Id.$$

Then, if $-1 \notin \sigma(\mathcal{L})$, $DF(0)$ is invertible on $\mathcal{H}$. By the local inverse function theorem, it is a bijection from an open neighbourhood $\mathcal{U}$ of ρ_{eq} to an open neighbourhood of 0. It follows that F never vanishes on $\mathcal{U} \setminus \{0\}$, then ρ_{eq} is an isolated equilibrium states for the norm of $L^2(\mathbb{R}^d;\, \mathrm{d}x/\rho_{eq})$. One could check that

$$\mathcal{T} : L^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d;\, \mathrm{d}x/\rho_{eq})$$

is continuous. Since it is the identity on the set of equilibrium states, then, ρ_{eq} is also isolated for the norm of $L^1(\mathbb{R}^d)$. ∎

We now turn to prove the proposition 5.4.5:

Proof of Proposition 5.4.5. Item i) is clear. Taking $\rho = \rho_2$ and $h = \rho_1 - \rho_2$ in (5.20), we first find

$$\mathcal{T}(\rho + h) \underset{h\to 0}{=} \mathcal{T}(\rho)\left(-W * h + \int_{\mathbb{R}^d} W * h\mathcal{T}(\rho)\,\mathrm{d}x + o(h)\right)$$

For all equilibrium state ρ_{eq}, we deduce ii):

$$D\mathcal{T}(\rho_{eq})(h) = -\rho_{eq} W * h + \rho_{eq}\int_{\mathbb{R}^d} W * h\rho_{eq}\,\mathrm{d}x = -\mathcal{L}(h).$$

By (5.11), we have for all h in $\mathcal{H}$,

$$\begin{aligned} D^2\mathscr{E}(\rho_{eq})(h,h) &= \int_{\mathbb{R}^d} h(h + \rho_{eq} W * h)\frac{\mathrm{d}x}{\rho_{eq}} + \left(\int_{\mathbb{R}^d} h\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^d} \rho_{eq} W * h\,\mathrm{d}x\right) \\ &= \int_{\mathbb{R}^d} h\left(h + \rho_{eq} W * h - \rho_{eq}\int \rho_{eq} W * h\,\mathrm{d}y\right)\frac{\mathrm{d}x}{\rho_{eq}} \\ &= \langle h, (Id + \mathcal{L})(h)\rangle_{\mathcal{H}}. \end{aligned}$$

iv) is also the consequence of a simple computation: if $\rho = \rho_{eq} + h$ is a solution to (5.1), then we have

$$\begin{aligned} \partial_t h &= \mathrm{div}(\nabla h + h\nabla(V + W * \rho_{eq}) + \rho_{eq}\nabla W * h + h\nabla W * h) \\ &= \mathrm{div}\left(\nabla\left(\frac{h + \rho_{eq} W * h}{\rho_{eq}}\right)\rho_{eq}\right) + \mathrm{div}(h\nabla W * h) \\ &= \mathrm{div}\left(\nabla\left(\frac{(\mathcal{L} + Id)(h)}{\rho_{eq}}\right)\rho_{eq}\right) + \mathrm{div}(h\nabla W * h). \end{aligned} \tag{5.29}$$

∎

The question to determine what can happen in the case where $\min(\sigma(\mathcal{L})) = -1$ is not solved yet. It is the most interesting case in the sense that it is what we expect from ρ_{eq} if it belongs to a non trivial attractive connected component of the set of the equilibrium states. In that case, for all h in $\ker(\mathcal{L} + Id)$ taking initial data $\rho_0 = \rho_{eq} + h$ in (5.1), we have by (5.29) and the definition of $\mathcal{L}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^d}(\rho-\rho_{eq})^2\frac{\mathrm{d}x}{\rho_{eq}(x)}\right]_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathbb{R}^d}\frac{\rho^2}{\rho_{eq}(x)}\,\mathrm{d}x - m\right]_{t=0}\\
&= \int_{\mathbb{R}^d}\frac{2\rho\partial_t\rho}{\rho_{eq}}\,\mathrm{d}x\\
&= -2\int_{\mathbb{R}^d}\frac{\rho_{eq}+h}{\rho_{eq}}\mathrm{div}(h\nabla W * h)\,\mathrm{d}x\\
&= 2\int_{\mathbb{R}^d}|\nabla W * h|^2 h\,\mathrm{d}x.
\end{aligned}$$

Changing h in $-h$ if necessary, it will be non negative on the whole set $\{\rho_{eq} + \lambda h \mid \lambda \geq 0\}$. If it is not equal to zero, we could expect ρ_{eq} to not be attractive, which doesn't mean that its connected component is not.

5.5 Appendix

The Compactness lemma

The proof of lemma 5.2.3 is classical, we put it here for sake of completeness.
Proof of lemma 5.2.3. According to **(H0)**, V is bounded from below. Without loss of generality, we suppose that it is non negative.

First step: uniform bounds
Owing to **(H0)**, it makes sense to introduce $a = \int_{\mathbb{R}^d} e^{-(1-\varepsilon)V}\,\mathrm{d}x$. Take ρ in $E_{m,r}$, we decompose its entropy as follows:

$$\mathscr{E}(\rho) = \int_{\mathbb{R}^d}\rho\ln\left(\frac{\rho}{a^{-1}e^{-(1-\varepsilon)V}}\right)\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^d}\rho V\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}\rho W*\rho\,\mathrm{d}x - m\ln(a)$$

The first three terms are non negative; therefore, we have:

$$\int_{\mathbb{R}^d}\rho\ln\left(\frac{\rho}{a^{-1}e^{(1-\varepsilon)V}}\right)\,\mathrm{d}x,\quad \varepsilon\int_{\mathbb{R}^d}\rho V\,\mathrm{d}x,\quad \frac{1}{2}\int_{\mathbb{R}^d}\rho W*\rho\,\mathrm{d}x \leq r + m\ln(a) \tag{5.30}$$

Splitting differently the entropy, we have more simply:

$$\int_{\mathbb{R}^d}\rho\ln(\rho)\,\mathrm{d}x \leq r \tag{5.31}$$

Second step: relative compactness
Take $R > 0$, for all ρ in $E_{m,r}$ we have:

$$\int_{|x|\geq R}\rho\,\mathrm{d}x \leq \left(\min_{|x|\geq R} V(x)\right)^{-1}\int_{\mathbb{R}^d}\rho V\,\mathrm{d}x \leq \left(\min_{|x|\geq R} V(x)\right)^{-1}\frac{r+\ln(a)}{\varepsilon}\xrightarrow[R\to\infty]{} 0$$

It is clear that under **(H0)**, $(1+V)e^{-(1+V)}$ belongs to $L^1(\mathbb{R}^d)$. Take A, a measurable subset of $\mathbb{R}^d$ with finite measure and $R>1$. For all ρ in $E_{m,r}$, we have:

$$\begin{aligned}
\int_A \rho\,\mathrm{d}x &\le \frac{1}{\ln(R)}\int_{A\cap\{\rho(x)\ge R\}}\rho\ln(\rho)\,\mathrm{d}x + \int_{A\cap\{\rho(x)\le R\}}\rho\,\mathrm{d}x\\
&\le \frac{1}{\ln(R)}\left(\int_{\mathbb{R}^d}\rho\ln\rho\,\mathrm{d}x - \int_{\{\rho(x)\le 1\}}\rho\ln(\rho)\,\mathrm{d}x\right) + R|A|\\
&\le -\frac{1}{\ln(R)}\left(\int_{\{0\le\rho(x)\le e^{-(1+V(x))}\}}\rho\ln\rho\,\mathrm{d}x\int_{\{e^{-(1+V(x))}\le\rho(x)\le 1\}}\rho\ln\rho\,\mathrm{d}x\right) + \frac{r}{\ln(R)} + R|A|\\
&\le \frac{1}{\ln(R)}\left(\int_{\mathbb{R}^d}(1+V)e^{-(1+V)}\,\mathrm{d}x + \int_{\mathbb{R}^d}\rho(1+V)\,\mathrm{d}x\right) + \frac{r}{\ln(R)} + R|A|\\
&\le \frac{B}{\ln(R)} + R|A|
\end{aligned}$$

where we have set up $B = r + \int_{\mathbb{R}^d}(1+V)e^{-(1+V)}\,\mathrm{d}x + m + (r+\ln(a))/\varepsilon$, and used (5.31) and (5.30). If $|A|<1$ taking $R = \left(\sqrt{|A|}\right)^{-1}$ we finally find for all ρ in $E_{m,r}$:

$$\int_A \rho\,\mathrm{d}x \le -\frac{2B}{\ln(|A|)} + \sqrt{|A|} \xrightarrow[|A|\to 0]{} 0.$$

Finally by the Dunford-Pettis Theorem $E_{m,r}$ is weakly relatively compact in $L^1(\mathbb{R}^d)$.

Closedness

Let $(\rho_k)_k$ be a sequence of elements of $E_{m,r}$ such that ρ_k goes to ρ weakly on $L^1(\mathbb{R}^d)$. Take $b = \int_{\mathbb{R}^d} e^{-V}\,\mathrm{d}x$. In order to prove that ρ belongs to $E_{m,r}$, we decompose the entropy as follows:

$$\mathscr{E}(\rho_k) = \int_{\mathbb{R}^d}\rho_k\ln\left(\frac{\rho_k}{mb^{-1}e^{-V}}\right)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}\rho_k W*\rho_k\,\mathrm{d}x - m\ln(b/m)$$

We prove that each term is lower semi-continuous for the weak topology. Let $\theta\in C(\mathbb{R}^d)$ be such that:

$$\theta(x)=1 \text{ for } |x|\le 1,\quad \theta(x)=0 \text{ for } |x|\ge 2,\quad 0\le\theta(x)\le 1 \text{ for any } x\in\mathbb{R}^d$$

For all $\epsilon>0$, we set:

$$W_\epsilon(x) = (1-\theta(x/\epsilon))W(x)\theta(\epsilon x).$$

For all $x\in\mathbb{R}^d$, since W_ϵ belongs to L^∞, we have:

$$\lim_{k\to\infty}\rho_k * W_\epsilon(x) = \rho * W_\epsilon(x)$$

Moreover, for $\epsilon>0$ fixed, W_ϵ is uniformly continuous on $\mathbb{R}^d$. Then we have:

$$\begin{aligned}
|W_\epsilon*\rho_k(x) - W_\epsilon*\rho_k(x')| &\le \int_{\mathbb{R}^d}|W_\epsilon(x-y)-W_\epsilon(x'-y)|\rho_k(y)\,\mathrm{d}y\\
&\le m\sup_{|x_1-x_2|\le|x-x'|}|W_\epsilon(x_1)-W_\epsilon(x_2)| \xrightarrow[|x-x'|\to 0]{} 0
\end{aligned}$$

Then ρ_k*W_ϵ goes to $\rho*W_\epsilon$ uniformly on all compact set of $\mathbb{R}^d$. It ensures:

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\rho_k(x)\theta(x/\epsilon)W_\epsilon*\rho_k(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\rho(x)\theta(x/\epsilon)W_\epsilon*\rho(x)\,\mathrm{d}x.$$

As ρ_k and W are non negative, we deduce that:

$$\liminf_{k\to\infty}\int_{\mathbb{R}^d}\rho_k(x)W*\rho_k(x)\,\mathrm{d}x \ge \int_{\mathbb{R}^d}\rho(x)\theta(x/\epsilon)W_\epsilon*\rho(x)\,\mathrm{d}x \qquad \text{holds.}$$

When ϵ goes to 0, we finally obtain by the monotone convergence theorem:

$$\liminf_{k\to\infty} \int_{\mathbb{R}^d} \rho_k(x) W * \rho_k(x)\, \mathrm{d}x \geq \int_{\mathbb{R}^d} \rho(x) W * \rho(x)\, \mathrm{d}x.$$

For the first term, a general proof (see [35, p.10-13]) allows us to ensure directly:

$$\liminf_{k\to\infty} \int_{\mathbb{R}^d} \rho_k \ln\left(\frac{\rho_k}{mb^{-1}e^{-V}}\right) \mathrm{d}x \geq \int_{\mathbb{R}^d} \rho \ln\left(\frac{\rho}{mb^{-1}e^{-V}}\right) \mathrm{d}x$$

Finally, we have proved:

$$\liminf_{k\to\infty} \mathscr{E}(\rho_k) \geq \mathscr{E}(\rho)$$

Then $E_{m,r}$ is closed.

■

A weaker version of the Lemma 5.4.4

When the set $Eq(m, \mathscr{E}^*)$ is finite, the convergence in Theorem 5.4.1−1) is easy to obtain (also using Theorem 5.3.1−1)). We point out that it is possible to generalise that idea asking the set $Eq(m, \mathscr{E}^*)$ to be countable. For a compact set, being countable is far more restrictive than being totally disconnected but the proof of that weaker result use a very different point of view from the one of the lemma 5.4.4, we put it here by sake of completeness.

Lemma 5.5.1 *Let X be a metric space and Y be a countable compact set of X. Let γ be a continuous function from $\mathbb{R}_+$ to X. Then the following assertion are equivalent:*

1. *The distance between $\gamma(t)$ and Y goes to 0 when t goes to infinity.*

2. *$\gamma(t)$ converges to some point of Y when t goes to infinity.*

Proof of lemma 5.5.1.
For all ordinal numbers $\beta \geq 0$, we define the β-th derivative set of Y by transfinite induction (see [52, section I.2] for basic notions on ordinal numbers and transfinite induction):

$$\begin{cases} Y_0 := Y & \\ Y_{\alpha+1} := \mathrm{Ac}(Y_\alpha) & \text{for all successor ordinal } \alpha+1 \\ Y_\beta := \bigcap_{\alpha<\beta} Y_\alpha & \text{for all limit ordinal } \beta \end{cases}$$

where for all subsets Z of X, $\mathrm{Ac}(Z)$ is the set of accumulation points of X:

$$\mathrm{Ac}(Z) := \{x \in X \quad | \quad \forall \epsilon > 0, \quad Z \cap (B(x,\epsilon) \setminus \{x\}) \neq \emptyset\}.$$

This sequence has been first defined by Cantor in [20] when he worked on trigonometric series (see also [23] or [72]).
If we suppose that $d(\gamma(t), Y)$ goes to 0, and that $\gamma(t)$ does not converge when t goes to infinity, then we prove by transfinite induction that for all ordinal numbers β, $d(\gamma(t), Y_\beta)$ also goes to 0 when t goes to infinity (in the end, we will prove that we can find λ such that Y_λ is empty which will give the contradiction).
For $\beta = 0$, it is obviously satisfied.

Fix an ordinal number β and suppose that it is true for all $\alpha < \beta$. For all $0 \leq \alpha < \beta$, we define $D_\alpha = Y_\alpha \setminus Y_{\alpha+1}$. For all $x \in D_\alpha$, we set:

$$\delta_x = \frac{1}{5} d(x, Y_\alpha \setminus \{x\})$$

which is always positive because x is not an accumulation point of Y_α. Whenever β is a limit or a successor ordinal, we have:

$$Y = Y_\beta \cup \left(\bigcup_{0 \leq \alpha < \beta} D_\alpha \right)$$

Then, for all $\epsilon > 0$, we define the following covering of Y:

$$Y \subset \left(\bigcup_{y \in Y_\beta} B(y, \epsilon/2) \right) \cup \left(\bigcup_{0 \leq \alpha < \beta} \bigcup_{x \in D_\alpha} B(x, \delta_x) \right)$$

Using Borel-Lebesgue's characterisation of compactness, we can find n and m, $\{y_1, .., y_n\} \subset Y_\beta$, $\alpha_1, .., \alpha_m < \beta$ and $(x_1, .., x_m) \in \prod_{j=1}^m D_{\alpha_j}$ such that:

$$Y \subset \left(\bigcup_{i=1}^n B(y_i, \epsilon/2) \right) \cup \left(\bigcup_{j=1}^m B(x_j, \delta_{x_j}) \right). \tag{5.32}$$

In order to simplify the notation, we set $\delta_j = \delta_{x_j}$. For all $\delta > 0$, we define Y_α^δ with:

$$Y_\alpha^\delta = \{x \in X \quad | \quad d(x, Y_\alpha) < \delta\}.$$

According with the hypothesis of induction, we can find $\eta \geq 0$ such that $I_\eta = \{\gamma(t) \mid t \geq \eta\} \subset Y_{\alpha_j}^{2\delta_j}$ for all $j \in \{1, .., m\}$. Suppose that for some j in $\{1, .., m\}$, $I_\eta \cap B(x_j, 2\delta_j) \neq \emptyset$. δ_j has been chosen such that $d(B(x_j, 2\delta_j), Y_{\alpha_j} \setminus \{x_j\}) \geq 3\delta_j$, then we have the decomposition:

$$Y_{\alpha_j}^{2\delta_j} = B(x_j, 2\delta_j) \cup \{x \in X \quad | \quad d(x, Y_{\alpha_j} \setminus \{x_j\}) < 2\delta_j\}.$$

Those two sets are open and disjoint. As I_η is a connected subset of $Y_{\alpha_j}^{2\delta_j}$, we have $I_\eta \subset B(x_j, 2\delta_j)$. As $d(\gamma(t), Y_{\alpha_j})$ goes to 0, $\gamma(t)$ goes to x_j when t goes to infinity which is in contradiction with the hypothesis. For all $j \in \{1, .., m\}$, and for all $t \geq \eta$, we have proven that $d(\gamma(t), x_j) \geq 2\delta_j$. Taking $\delta = \min_{1 \leq j \leq m} \delta_j$, for all $t \geq \eta$, we have:

$$d\left(\gamma(t), \bigcup_{j=1}^m B(x_j, \delta_j) \right) \geq \delta.$$

Then, using (5.32), we have:

$$d\left(\gamma(t), \bigcup_{i=1}^n B(y_i, \epsilon/2) \right) \xrightarrow[t \to \infty]{} 0.$$

It allows us to find η' such that for all $t \geq \eta'$, $d(\gamma(t), \{y_1, .., y_n\}) < \epsilon$. It ends the induction and for all ordinal numbers β, we have:

$$d(\gamma(t), Y_\beta) \xrightarrow[t \to \infty]{} 0.$$

It is well known that all complete spaces $Z \neq \emptyset$ such that $Z = \mathrm{Ac}(Z)$ are not countable. Therefore, as Y is countable, if $Y_\alpha = Y_{\alpha+1}$, then $Y_\alpha = \emptyset$. Suppose now that for all ordinal numbers α, $Y_\alpha \neq \emptyset$. Then, for all α, we can find z_α in $Y_\alpha \setminus Y_{\alpha+1}$. The ordinal number theory

ensures that we can find an ordinal number ω_1 such that the set $\{\alpha < \omega_1\}$ is not countable. The mapping $\alpha \mapsto z_\alpha$ from the set $\{\alpha < \omega_1\}$ to Y is injective. Then, Y can not be countable which is in contradiction with the hypothesis. We set λ, the first ordinal number such that $Y_\lambda = \emptyset$.
We have proven that as soon as $\gamma(t)$ does not converge, if the distance between $\gamma(t)$ and Y goes to 0 when t goes to infinity, so does the distance between $\gamma(t)$ and Y_α for all α. Taking $\alpha = \lambda$, we find:

$$d(\gamma(t), \emptyset) \xrightarrow[t \to \infty]{} 0$$

which is wrong.

$\blacksquare$

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